Complex Dynamics

Complex Numbers:

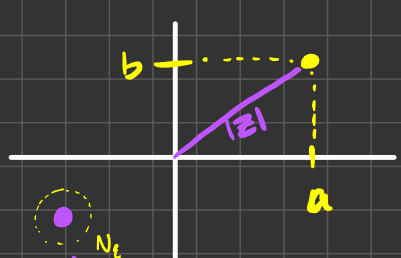
Let $i = \sqrt{-1}$ be a number which formally satisfies $i^2 = -1$.
A complex number is an expression of the form
$a + bi$ where $a, b \in \mathbb{R}$, $i = \sqrt{-1}$

These numbers are a field:

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i$$

$$(a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + a_2 b_1 i - b_1 b_2$$

Every complex $z = a + bi$ has a complex conjugate
$\bar{z} = a - bi$ and $z \cdot \bar{z} = (a+bi)(a-bi) = a^2 + b^2 = |z|^2$

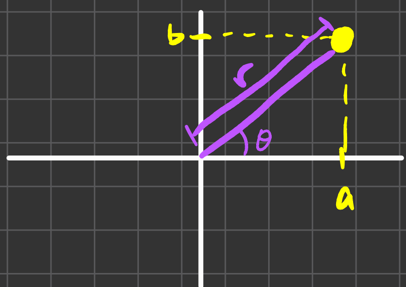


This notion can be used to give the complex plane a metric:

$$d(z_1, z_2) = |z_1 - z_2|$$

So, the Complex plane is a metric space, so we have notions of neighborhoods, convergence, continuity, etc.

$$N_\varepsilon(w) = \{z \in \mathbb{C} : |z-w| < \varepsilon\} = B_\varepsilon(w)$$



$$z = a+bi = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$e^{i\theta} = \cos\theta + i\sin\theta = ze^{i\theta}$$

Euler's Identity

r is the modulus of z

θ is the argument of z

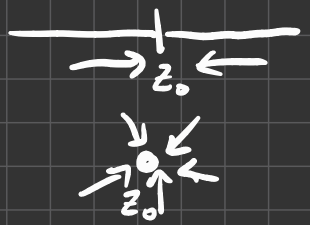
$\theta = \arg(z)$

$$a = |z|\cos\theta$$
$$b = |z|\sin\theta$$

Functions

A function $f: \mathbb{C} \to \mathbb{C}$ is continuous if it is continuous from the metric space defn.

A function $f: \mathbb{C} \to \mathbb{C}$ is differentiable at z_0 if $\lim\limits_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ exists independently of direction



We denote this as $f'(z_0)$

If f is differentiable at every pt in an open set $D \subset \mathbb{C}$, then it is differentiable over D

If f is differentiable over D, then it is continuous over D.

Properties of Complex Derivatives

If f, g are differentiable over D:

(1) $(f+g)' = f' + g'$

(2) $(f \cdot g)' = f'g + fg'$

(3) $(f \circ g)' = f' \circ g\,(g')$

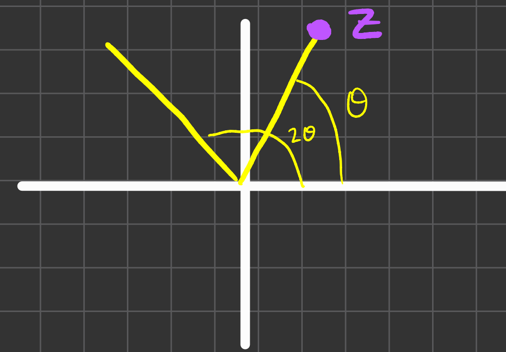
(4) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ at the points where $g \neq 0$

More Properties

$f(z) = z^n \implies f'(z) = nz^{n-1}$

$z = |z|e^{i\theta} \rightarrow z^2 = |z|^2 e^{i\theta 2}$

$z = |z|e^{i\theta} \rightarrow z^n = |z|^n e^{i\theta n}$



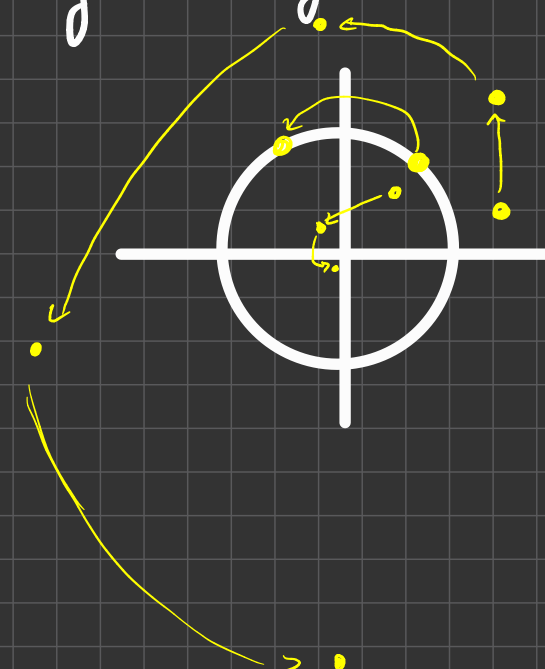
If z is inside unit circle, the iterations $z, z^2, z^4, \ldots, z^{2n}$ spiral towards 0

If z is on the unit circle, the iterations jump around the unit circle

If z is outside the unit circle, the iterations spiral away from origin

$W^s(0) = N_1(0)$ ⟵ All points in $\mathbb{C}$ w/ $|z| < 1$

$W^s(\infty) = \{z \in \mathbb{C} : |z| > 1\}$



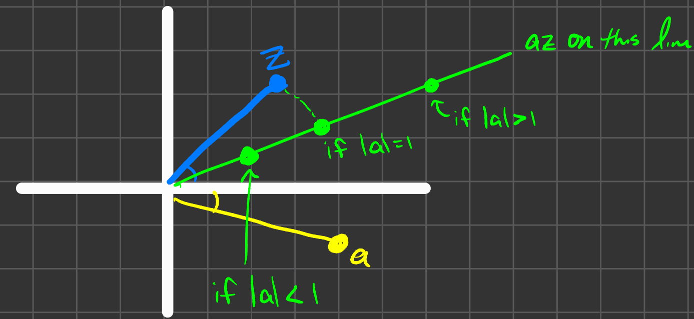
On the unit circle, parameterized by $\theta \in [0, 2\pi)$

$\theta \mapsto 2\theta$

Angle Doubling Map

Ex2:

$$f(z) = az$$
$$= |a| e^{i\arg(a)} \cdot |z| e^{i\arg(z)}$$
$$= |a||z| e^{i(\arg(a)+\arg(z))}$$

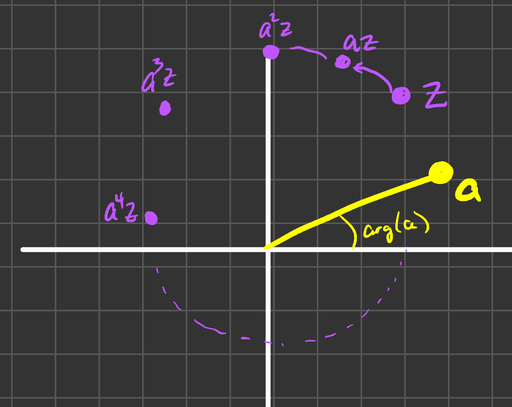


$$f^n(z) = a^n z = |a|^n |z| e^{i(n\arg(a)+\arg(z))}$$

If $|a|<1$: $W^s(0) = \mathbb{C}$

If $|a|>1$: $W^s(\infty) = \mathbb{C} \setminus \{0\}$

If $|a|=1$: $f^n(z) = |z| e^{i(\arg(a)n + \arg(z))}$
$$= z \cdot e^{i n \arg(a)}$$



There are 2 cases:

(1) If $\arg(a) = \frac{p}{q}\pi$, $p, q \in \mathbb{N}$:

Then $f^{2q}(z) = z$ for any z $\Rightarrow$ every point is periodic

2) If $\arg(a) \neq \frac{p}{q}\pi$, $p, q \in \mathbb{N}$ e.g. $\arg(a) = \sqrt{2}\pi$

Then there are no periodic orbits other than 0

In this case, the orbit of z is dense in the circle of radius $|z|$

If $f: \mathbb{C} \to \mathbb{C}$ is a differentiable function and p is fixed by p, there is $|f'(p)| < 1$, then the stable set of p contains a neighborhood of p. If $|f'(p)| > 1$, then there is a neighborhood of p which moves away from p.

$|f'(p)| < 1$: Attracting Fixed Point

$|f'(p)| > 1$: Repelling Fixed Point

Day 2

We want to look at dynamics of quadratic polynomials

$f(z) = z^2 + az + b$

We proved that all of these are conjugate to fctns of the form $q_c(z) = z^2 + c$ (Old GW)

Last time we looked at $q_0(z) = z^2$

If $|z| < 1$, then $q_0^n(z) \longrightarrow 0$

$|z| > 1$, then $|q_0^n(z)| \longrightarrow \infty$

$|z| = 1$, then Doubling Map

Prop: The orbit of a pt under a quadratic polynomial is either bounded or in the stable set of ∞

Proof (Outline):

1) It sufficies to prove this for the family $q_c(z) = z^2 + c$

2) Use a triangle inequality & induction to prove that if $|w| > |c|+1$ then, $w \in W^s(\infty)$

Def: The filled Julia Set of q_c is the set of points which are bounded, denoted by K_c

$K_c = \{z \in \mathbb{C} : \exists M \text{ s.t. } |q_c^n(z)| < M, \forall n\}$

Def: The Julia Set is the boundary of K_c, denoted J_c

Aka points in K_c s.t. any neighborhood spills out.

So for $q_0(z) = z^2$:

If $|z| < 1$, then $q_0^n(z) \to 0 \Rightarrow K_c = W^s(0)$

$|z| > 1$, then $|q_0^n(z)| \to \infty$

If $|z| = 1$, then doubling map. $\Rightarrow J_c$

Fact: If a complex polynomial has an attracting periodic orbit, there must be a critical point of the polynomial in the stable set of one of the points in the orbit

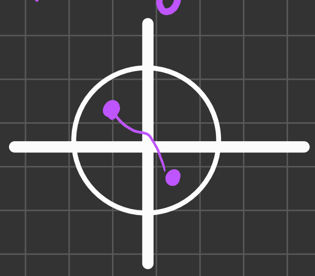
Def: Critical point of $p(z)$ is a z_0 s.t. $p'(z_0) = 0$

For $q_c(z)$, $z=0$ is the only critical pt.

Theorem: If $0 \in K_c$, then K_c is connected.
If $0 \notin K_c$, then K_c is a Cantor set

K_c is connected if for any $x, y \in K_c$, there is a path connecting x, y that is completely contained in K_c

Ex:



There is a path contained in the odd

Cantor Set is a closed, bounded, totally disconnected set with its isolated pts.

We want to know more about the set:

$$M = \{c : q_c^n(0) \text{ is bounded}\}$$
$$= \{c \in \mathbb{C} : 0 \in K_c\}$$

You can show that $M \subset B_2(0)$ (ball of radius 2 around origin)

In other words, if $|c| > 2$, then $q_c^n(0)$ is unbounded.

If we define $M_k = \{c : q_c(z) \text{ has an attracting periodic orbit of prime period } k\}$

$$\bigcup_{k=1}^{\infty} M_k \subset M$$

M is the <u>Mandelbrot Set</u>

Look at $r_c(z) = z^3 + c$

You can define the filled Julia Sets the same way and

Consider $M^{(3)} = \{c : r_c^n(0) \text{ is bdd}\}$

$$M^{(k)} = \left\{c : \begin{array}{c}\text{orbit of } 0 \text{ under } z \mapsto z^k + c \\ \text{is bounded}\end{array}\right\} = \underline{\text{Multibrot Set}}$$

Ramblings

Pick any $\bar{x} \in \Sigma_2^+$.

$$q_c^n(0) = \underbrace{q_c \circ q_c \circ \dots \circ q_c}_{n\text{-times}}(0) \leftarrow \text{gives } M$$

$$q_c^{(0)}(z) = z^2 + c$$

$$q_c^{(1)}(z) = z^3 + c$$

$$M^{(\bar{x})} = \left\{c : \begin{array}{c} q_c^{(x_n)} \circ \dots \circ q_c^{(x_2)} \circ q_c^{(x_1)}(0) \\ \text{is bounded}\end{array}\right\}$$

$$\bar{q} = q_c^{(0)} \circ q_c^{(1)}$$

polynomial of degree J