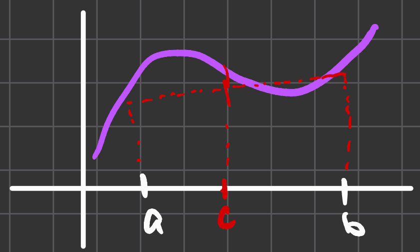
Differentiability

Recall: A function $f: I \to \mathbb{R}$ (Interval) is differentiable @ $x \in I$ if $f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$ exists. If f is differentiable at every $x \in I$, then it is differentiable on I

Mean Value Theorem: Sps $f: [a,b] \to \mathbb{R}$ is differentiable on I and f is continuous, then there is a $c \in (a,b)$ s.t.

$$f(b) - f(a) = f'(c)(b-a)$$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



Thm: Let $f: I \to \mathbb{R}$ be differentiable, $f(I) \subseteq I$, $|f'(x)| < 1$ for all $x \in I$, f' is continuous. Then, there is a unique fixed point and $\underbrace{|f(x) - f(y)| < |x - y|}_{\text{Contraction}}$

For any $x, y \in I \Rightarrow$ Mapping brings pts closer together.

Proof: By the MVT, $|f(x)-f(y)| = |f'(c)||x-y| < |x-y|$

So, its a contraction

We already know there is a fixed pt p. Sps q is another fixed pt.

$$\underbrace{|p-q| =}_{p \,\&\, q \text{ are fixed}} |f(p)-f(q)| \underbrace{<}_{\text{By contraction}} |p-q|$$

So q cannot be fixed. Therefore, p is the unique fixed pt.

Note: If $|f'(x)| < \lambda < 1$ for all $x \in I$, then $|f(x)-f(p)| < \lambda|x-p|$. ($p$ is fixed by f)

$|f^2(x)-f^2(p)| < \lambda|f(x)-f(p)| < \lambda\lambda|x-p|$

$|f^3(x)-f^3(p)| < \lambda|f^2(x)-f^2(p)| < \dots < \lambda^3|x-p|$

$\Rightarrow \boxed{W^s(p) \supseteq I}$

Ex: $f(x) = rx+b$, $|r|<1$

$f'(x) = r$, $f(p) = rp+b = p \Rightarrow p = \frac{b}{1-r}$ is fixed pt.

$W^s(p) = \mathbb{R}$

$f^n(x) \to p$ exponentially in n

$|f^n(x)-p| < |r|^n|x-p|$

Ex: $f(x) = rx + b$, $|r| = 1$

Not a contraction.

* If $r = 1$ & $b = 0$, everything is fixed

$b \neq 0$, no fixed pts

* If $r = -1$, there is a fixed pt @ $p = \frac{b}{2}$

So, $W^s(p) = \{p\}$

$f^2(x) = -(-x + b) + b$
$= x - b + b = x$

So, every pt is periodic of period 2.

Ex: $f(x) = rx + b$, $|r| > 1$

There is a fixed pt at $p = \frac{b}{1-r}$

$|f(x) - p| = |f(x) - f(p)| = |rx + b - rp - b| = |r(x-p)|$

since $|r| > 1$, $|r(x-p)| > |x-p|$

So $W^s(p) = \{p\}$

and $W^s(\infty) = \mathbb{R} - \{p\}$

Ex: $f(x) = x^2$ ... $f(0) = 0$, $f(1) = 1$

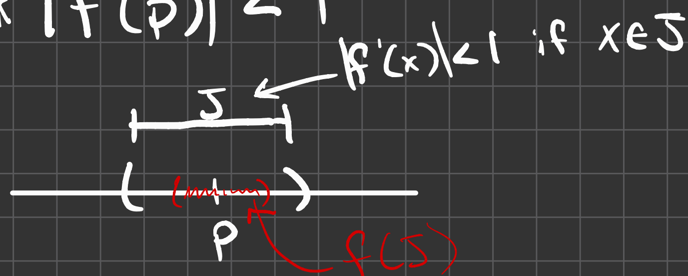
$f'(x) = 2x$ $\quad W^s(0) = (-1, 1)$

If $x \in (-\frac{1}{2}, \frac{1}{2})$, $|f'(x)| < 1$, so $(-\frac{1}{2}, \frac{1}{2}) \subset W^s(0)$

Thm: Let f be differentiable on I, p is fixed by f, $|f'(p)|<1$, and f' continuous on I, then there is an interval $J \subset I$ containing p such that $J \subset W^s(p)$

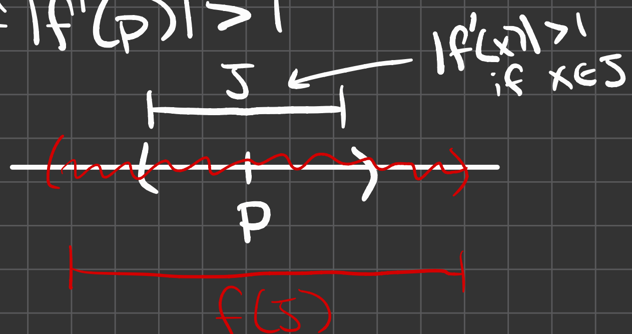
* If $|f'(p)|>1$, then there is an interval $J \subset I$ containing p such that $x \neq p$ leaves J eventually

Proof: * $|f'(p)|<1$



f is contracting on J, so $J \subset W^s(p)$

* if $|f'(p)|>1$



After every iteration, x is pushed further away from p, eventually leaves J (It could come back depending on mapping)

Def: If $f(p)=p$, then p is:

- **Hyperbolic**: if $|f'(p)| \neq 1$
- **Non-Hyperbolic**: if $|f'(p)| = 1$
- **Attracting**: if $|f'(p)| < 1$
- **Repelling**: if $|f'(p)| > 1$

So, you can make conclusions about the behavior of a fctn based on info of the derivative

Ex: $f(x) = x - x^3$

$f(0)=0$. $f'(x) = 1-3x^2$.

$f'(0)=1$, but if x is very close to 0, $|f'(x)| < 0$ and this gives a contraction and you can conclude that $W^s(0) \subset B_r(0)$ for some r.

Ex: $f(x) = x + x^3 \Rightarrow f'(0) = 1,\ f'(x) = 1 + 3x^2$

When x is close to 0, $|f'(x)| > 1$, so points are being repelled from 0

Def: Let f be C^1 = set of fctns w/ continuous derivatives and p a periodic point of period K.

Then if $|(f^K)'(p)| \neq 1$, then p is a hyperbolic periodic point

$|(f^K)'(p)| < 1$ is attracting

$|(f^K)'(p)| > 1$ is repelling

Corollary: If p is periodic w/ period K, and:

* If $|(f^K)(p)| < 1$, then there is an interval J containing p s.t. $J \subset W^s(p)$
* If $|(f^K)'(p)| > 1$, then p is contained in an interval J of points which eventually leave J