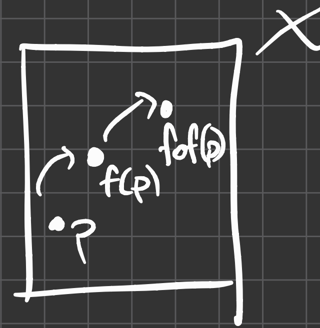
Discrete Dynamical Systems

$$\underset{\substack{\text{State space}\\ \text{Phase space}}}{\overset{\text{set}}{X}} \xrightarrow[\text{rule}]{f} X$$



$$p \in X \qquad f(p) \qquad f\circ f(p)$$

$$\boxed{\underbrace{f\circ \cdots\cdots \circ f}_{n \text{ times}} = f^n(p)} \longleftarrow \begin{array}{l}\text{n-fold composition}\\ \text{n units of time}\end{array}$$

Ex: $X = \mathbb{R}$ $\qquad f(x) = -x^2$

$$f\circ f(x) = f^2(x) = -x^4$$

$$f^3(x) = -x^8$$

$$f^n(x) = -x^{2^n}$$

What do we want to know?

What is the long term behavior of f?

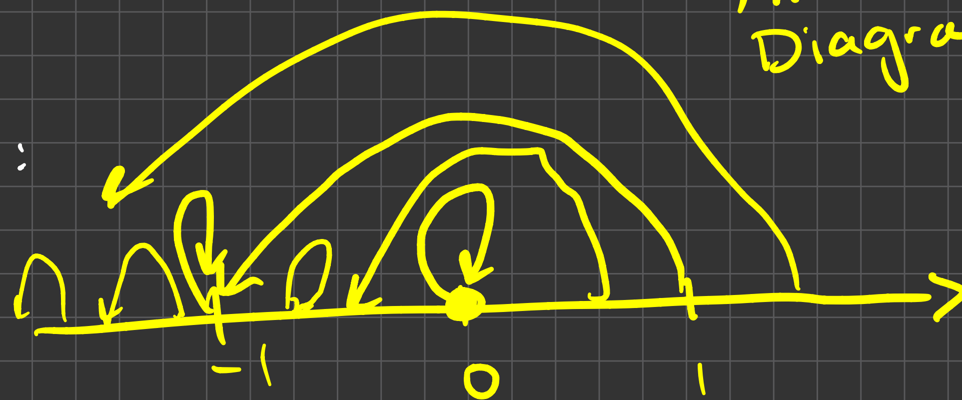
How does it depend on starting point/condition?

Does the $\lim_{n\to\infty} f^n(p)$ exist?

For the example $f(x) = -x^2$:

$$\lim_{n\to\infty} f^n(x) = \begin{cases} 0 & \text{if } x \in (-1,1) \\ -1 & \text{if } x = \pm 1 \\ -\infty & \text{if } |x| > 1 \end{cases}$$

Phase Diagram



Population Models

Pretend the population of rabbits increases by 1.05% every week
How many are there after n weeks?

$$f(p) = 1.015p \to f \circ f(p) = (1.015)^2 p \to f^n(p) = (1.015)^n p$$

$$\lim_{n \to \infty} f^n(p) = \infty \text{ for all } p > 0$$

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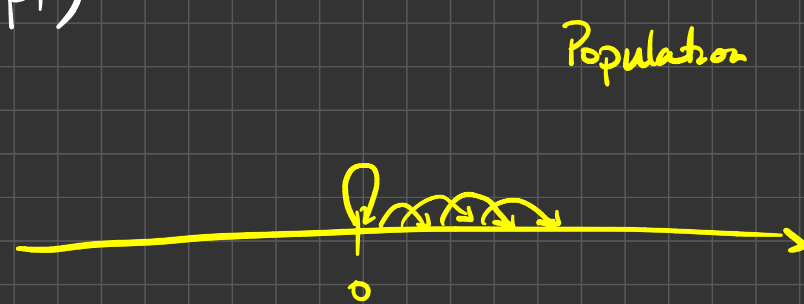
p_1 = population of rabbits $\quad$ p_2 = population of foxes

$$f(p_1, p_2) = (1.015p_1 - .005p_2,\ 1.005p_2 + .0001p_1)$$

$$f\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1.015 & -.005 \\ .0001 & 1.005 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$f^n\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1.015 & -.005 \\ .0001 & 1.005 \end{pmatrix}^n \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

f has eigenvalues $\to$ $\lambda_1 \approx 1.01495$, $\lambda_2 \approx 1.005$

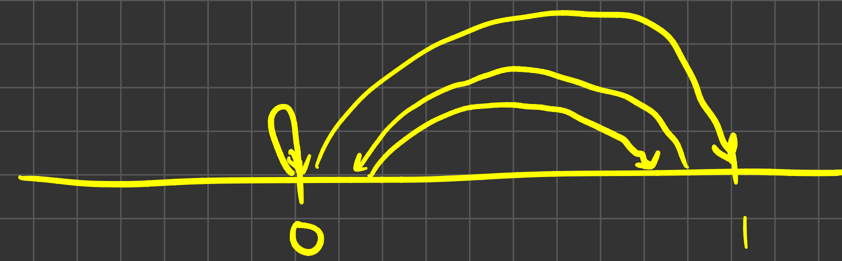


The eigendata determines the long term behavior of the system

Logistic Model

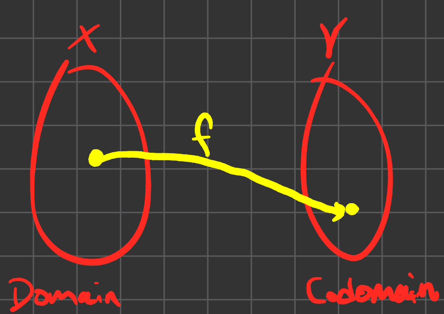
↪ prevents population explosion

$$f(p) = rp(1-p)$$
$$= rp - rp^2$$



Functions

A function is a rule that assigns an element of one set X an unique element of set Y



X is the domain of f

Y is the codomain of f

$$f: X \to Y$$

Functions are also called maps/mappings

The **Range** of f is the set of points in Y which come from X.

$$\text{Range}(f) = \{y \in Y : y = f(x) \mid x \in X\} = f(X)$$

If $A \subseteq X$, then the set **Image** of A is:

$$f(A) = \{y \in Y : y = f(x) \text{ for some } x \in A\}$$

If you have a set $B \subseteq Y$, the inverse image or preimage of B is

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

This is always defined. f does not need to have an inverse for this to make sense

$f(x) = x^2$

$f((-1,1)) = [0,1)$

$f^{-1}(\{4\}) = \{-2,2\}$

A function is one to one or injective if each element of the domain gets assigned a unique element of the range

$f(x) = f(y)$ iff $x = y$

Check via horizontal line test

$f(x) = x^3$ is injective

A function is onto or surjective if for any y in the codomain, there is a x assigned to it.

$$\text{Range}(f) = \text{Codomain}(f)$$

A function, $f: A \to B$ where $A, B \subseteq \mathbb{R}$, and $x \in A$ is continuous at x if for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $x' \in A$ and $|x' - x| < \delta$ then $|f(x) - f(x')| < \varepsilon$