Parameterized Families

Ex: $f_r(x) = rx$ ← Parameter

0 is the only fixed pt $\Longrightarrow$ 0 is $\begin{cases} \text{attracting if } |r|<1 \quad W^s(0) = \mathbb{R} \\ \text{repelling if } |r|>1 \quad W^s(0) = \{0\} \\ \text{one of infinitely many fixed pts if } |r|=1 \end{cases}$

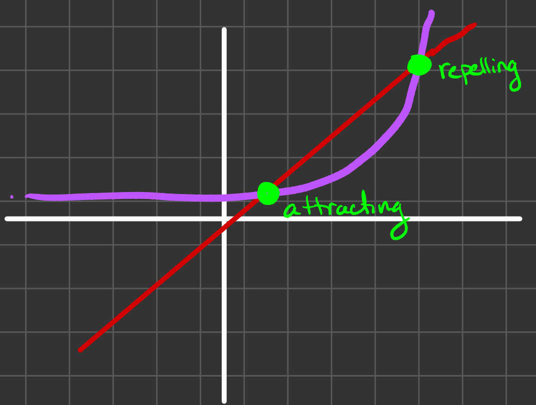
$f_r'(0) = r$

Drastic Change In Behavior of Crossing $r = \pm 1$ is a **Bifurcation**

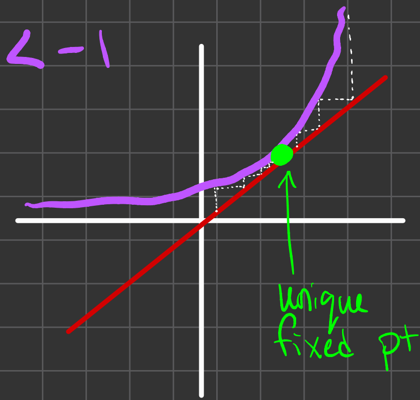
Def: Let $f_r(x)$ be a parameterized family of fctns. Then, there is a **Bifurcation** at r_0 if there is an $\varepsilon > 0$ s.t. whenever $a \in (r_0 - \varepsilon, r_0)$ and $b \in (r_0, r_0 + \varepsilon)$, then the dynamics of the maps f_a & f_b are "different."

Ex: $f_r(x) = e^{x+r}$

$r < -1$

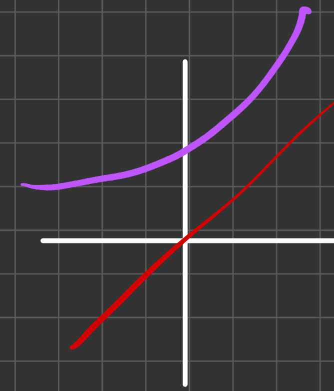


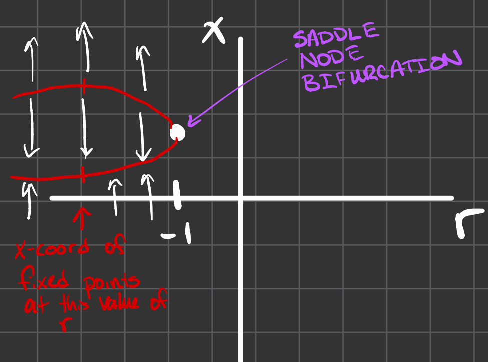
$r = -1$



$W^s(\text{unique fixed pt}) = (-\infty, pt]$

$r > -1$



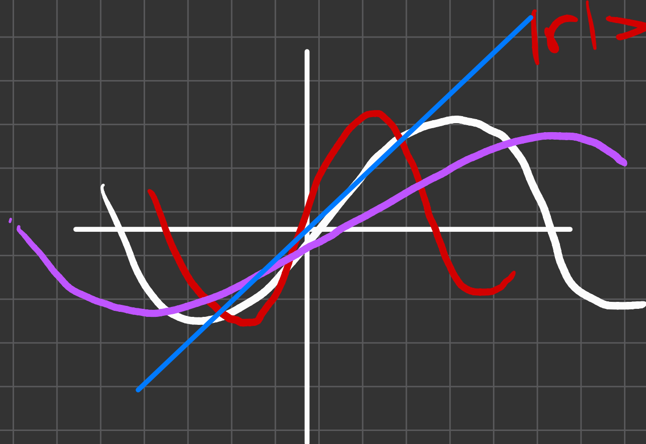


Ex: $f_r(x) = \sin(rx)$

$y = x$

$|r| < 1$ → Attracting fixed pt at 0

$|r| > 1$ ↪ could be either



Pitchfork Bifurcation

Ex: $f_r(x) = rx(1-x)$

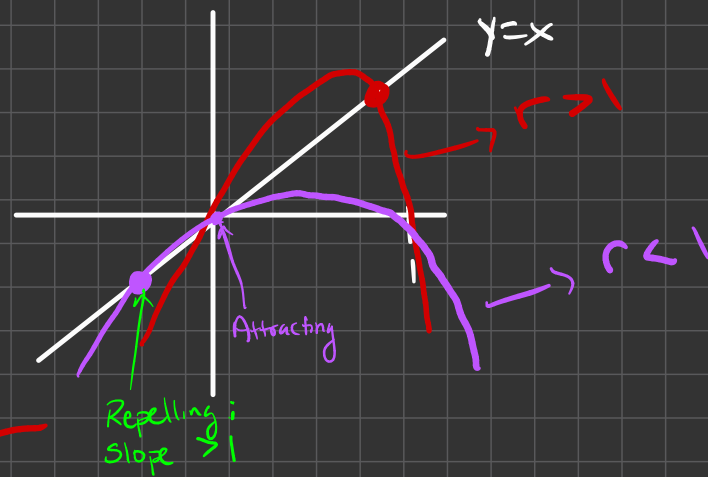
Fixed pts: $\{0, \frac{r-1}{r}\}$

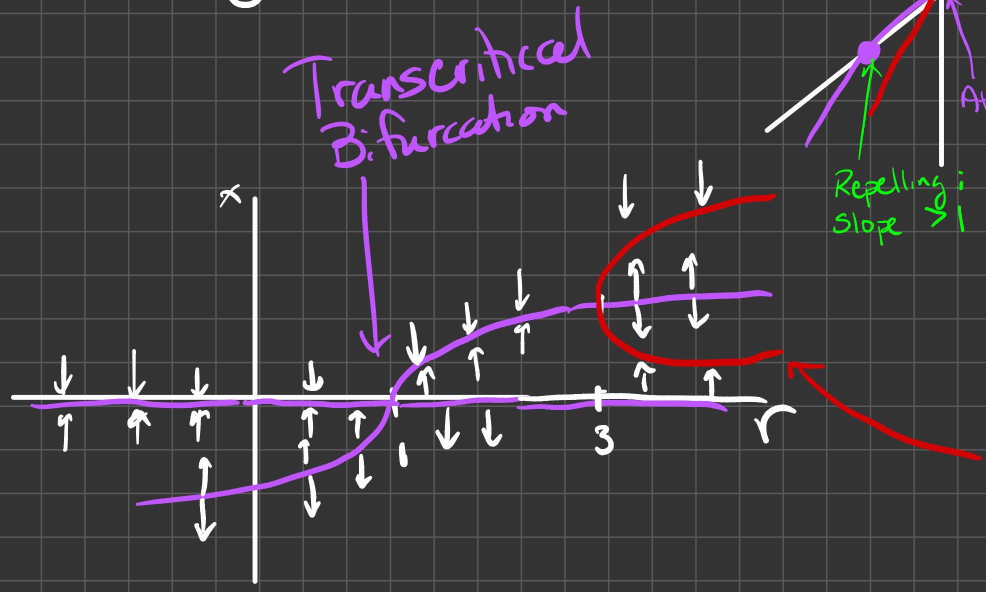
$f_r'(x) = r - 2rx$

$f_r'\left(\frac{r-1}{r}\right) = r - 2r\left(\frac{r-1}{r}\right)$

$= r - 2r + 2$

$= -r + 2$

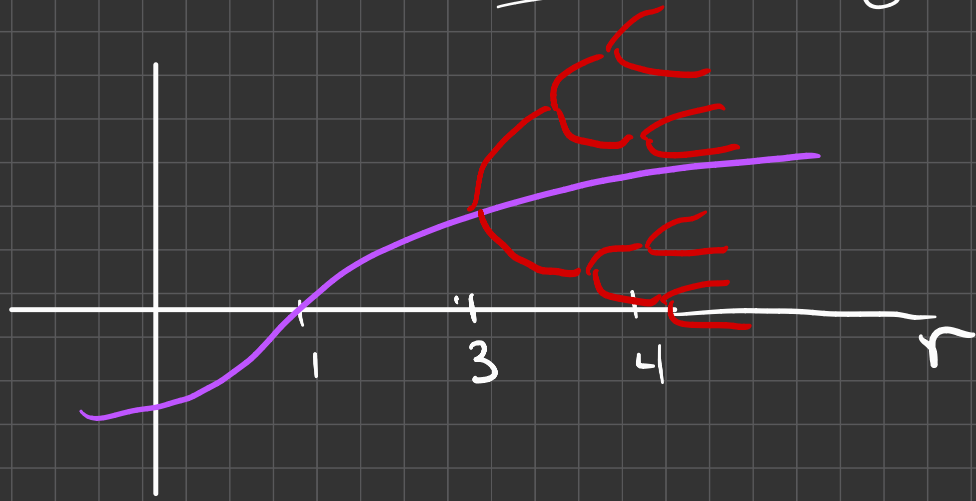




Main Types of Bifurcations:

- Saddle Node
- Pitchfork
- Transcritical
- Period Doubling

Period Doubling Cascade



What we Know of the Logistic Map:

1) Max: $\left(\frac{1}{2}, \frac{r}{4}\right)$

2) Fixed pts: $\left\{0, \frac{r-1}{r}\right\}$

$\left\{1, \frac{1}{r}\right\}$ are eventually fixed

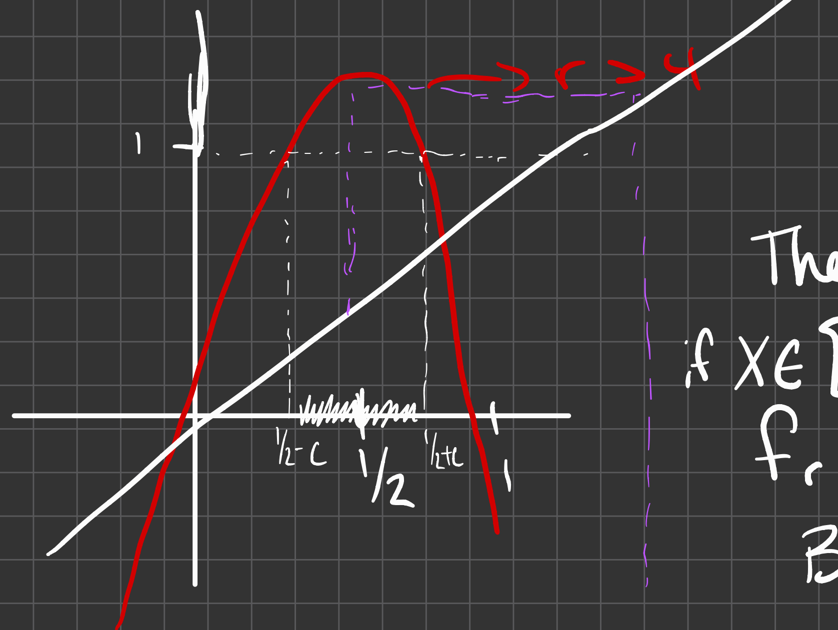
3) When $r=3$, period doubling bifurcation

4) When $r=1+\sqrt{6}$, period doubling

5) When $r = 1+\sqrt{8}$ there is a period 3 orbit $\Rightarrow$ Periodic orbit of every prime period

6) When considered as a fctn w/ domain $[0,1]$,
$$\mathrm{Im}\, f_r = f_r([0,1]) = [0, r/4]$$

y of vertex > 1 when r > 4



$f_r\left(\tfrac{1}{2}\right) > 1$ and $f_r^n\left(\tfrac{1}{2}\right) \to \infty$

There's a $c > 0$ s.t.

if $x \in [0, \tfrac{1}{2} - c] \cup [\tfrac{1}{2} + c, 1]$, then
$f_r(x) \in [0,1]$

But maybe $f^2(x) \notin [0,1]$

Define:
$$\Lambda_n^r = \{ x \in [0,1] : f_r^n(x) \in [0,1] \}$$

$\Lambda^r = \bigcap_{n>0} \Lambda^r_n$ = set of points that never leave

You consider $r>4$:

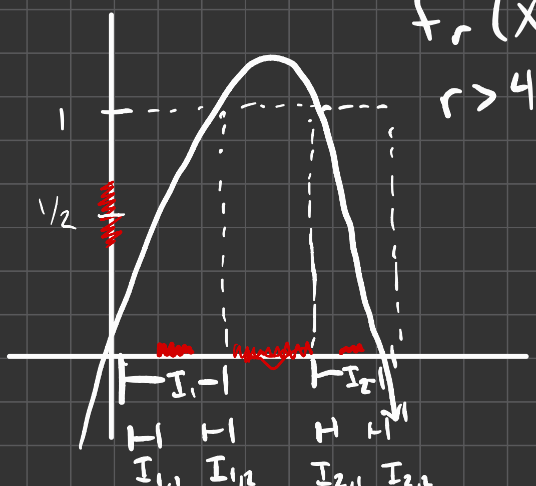
$f_r : \Lambda^r \longrightarrow \Lambda^r$

Next Week: What does Λ^r look like?

Next Class:

$f_r(x) = rx(1-x)$

$r>4$



If $\Lambda^r_n = \{x \in [0,1] : f^n_r(x) \in [0,1]\}$

= set of points that never leave after n iterations.

$\Lambda^r = \bigcap_n \Lambda^r_n$ = set of points that never leave

$\Lambda^r_0 = [0,1]$

$\Lambda^r_1 = I_1 \cup I_2$ where I_1 and I_2 are closed intervals.

Note: $f_r(I_1) = f_r(I_2) = [0,1]$

$\Lambda^r_2 = I_{1,1} \cup I_{1,2} \cup I_{2,1} \cup I_{2,2}$ Note: $I_{i,j} \subseteq I_i$

$$
\begin{tikzcd}[column sep=tiny]
 & & & & & & [0,1] \arrow[dlll, no head] \arrow[drrr, no head] & & & & & & & = \Lambda^r_0 \\
 & & & I_1 \arrow[dll, no head] \arrow[drr, no head] & & & & & & I_2 \arrow[dll, no head] \arrow[drr, no head] & & & & \Rightarrow \Lambda^r_1 \\
 & I_{1,1} \arrow[dl, no head] \arrow[dr, no head] & & & & I_{1,2} \arrow[dl, no head] \arrow[dr, no head] & & I_{2,1} \arrow[dl, no head] \arrow[dr, no head] & & & & I_{2,2} \arrow[dl, no head] \arrow[dr, no head] & & \Rightarrow \Lambda^r_2 \\
I_{1,1,1} & & I_{1,1,2} & & I_{1,2,1} & & I_{1,2,2} \; I_{2,1,1} & & I_{2,1,2} & & I_{2,2,1} & & I_{2,2,2} & \Rightarrow \Lambda^r_3 \\
\vdots & & \vdots & & \vdots & & \vdots \quad \vdots & & \vdots & & \vdots & & \vdots &
\end{tikzcd}
$$

Notice: $f(I_{2,2,2}) = I_{2,2}$

$f(I_{2,2}) = I_2$

$f(I_2) = [0,1]$

$\Lambda^r_n = \bigsqcup I_{i_1,\dots,i_n}$

$\uparrow$ Disjoint union of 2^n closed Intervals

$f^n(I_{a_1,\ldots,a_n}) = [0,1]$ one to one & onto

What does Λ^r look like?

Def: A non empty set $C \subseteq \mathbb{R}$ is a **Cantor Set** if it is

1. C is closed & bounded
2. C contains no intervals
3. Every point in C is an accumulation point of C

* Emphasis *

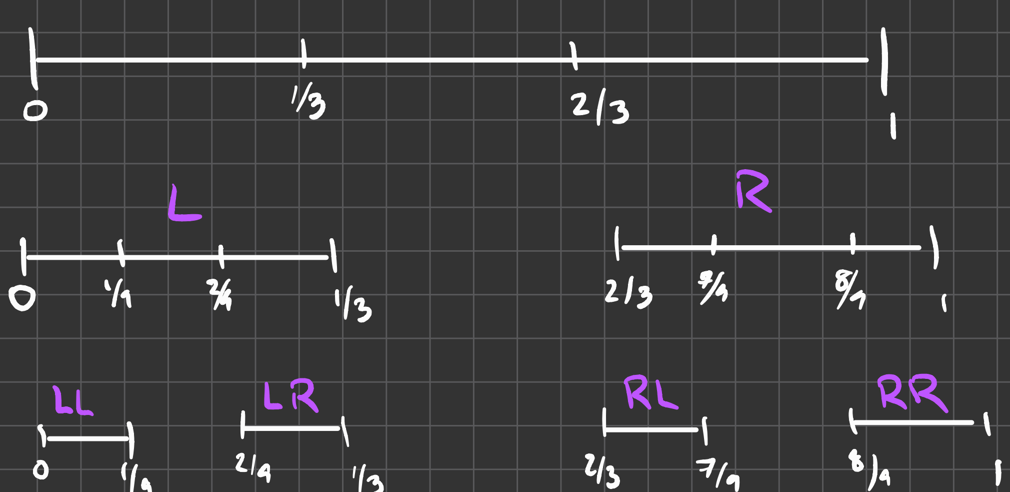
Ex: Cantor's Middle Third Set

C_0

C_1

C_2

$\vdots$



$x \in C, \ x = LRLLLR\ldots$

$C = \bigcap_{n > \infty} C_n$ ← Cantor Middle Third

Facts: • C_n is a disjoint union of 2^n intervals of length 3^{-n}

Total length of $C_n = \left(\frac{2}{3}\right)^n$

$\Rightarrow C$ contains no intervals

• The Compliment of C is open, so C must be closed & bounded

What's in C?

Each subinterval in C_n has an index in $\{L, R\}^n$

so points in $C = \bigcap_{n>0} C_n$ have addresses in $\{L, R\}^{\mathbb{N}}$

That is every point can be located by an infinite string of Ls & Rs

if $x = .s_1 s_2 s_3 s_4 \ldots,$

then $x_1 = .s_1 *$ $\qquad s_i \in \{L, R\}$

$x_2 = .s_1 s_2 *$

$\vdots$

$\vdots$

$x_n = .s_1 s_2 \ldots s_n *$

So, C is a Cantor set

An endpoint of C_n has an address of form:

Ex:

$2/9 = .LRLLL\ldots = .LRL^\infty$

So endpoints correspond to strings which are eventually all L's or R's

So the other type of addresses correspond to other points.

Back to Logistic Fctn

Prop: Let $r > 2+\sqrt{5} > 4$, $f_r(x) = rx(1-x)$

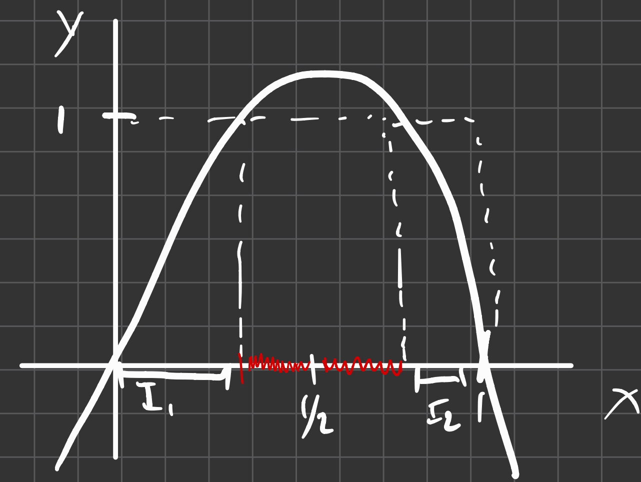
There is a $\lambda > 1$ s.t. $|f_r'(x)| > \lambda > 1$ for all $x \in \Lambda^r$

Also, the length of each $I_{z_1}, \ldots, I_{z_n}$ is less than λ^{-n}

Proof: It can be calculated that

$$|f_r'(x)| \geq \left|f_r'\left(\frac{1}{2} \pm \frac{\sqrt{r^2-4r}}{2r}\right)\right| > 1$$

endpoints of gap
if $r > 2+\sqrt{5}$



This proves the first part

Since $f(I_1) = [0,1]$

Since Length of $[0,1] = 1$ and the Stretching (derivative) on I_1 is at least λ, then I_1 has to have length $< \frac{1}{\lambda}$. Otherwise if length of $I_1 > 1/\lambda$, then $1 =$ length of $[0,1] \geq$ (length of I_1) $\lambda > 1$

By same argument, I_2 has length $\leq 1/\lambda$

Repeating the same argument, you get λ^{-n}

Thus, length of each $I_{2i,\ldots 2n}$ is less than λ^{-n}

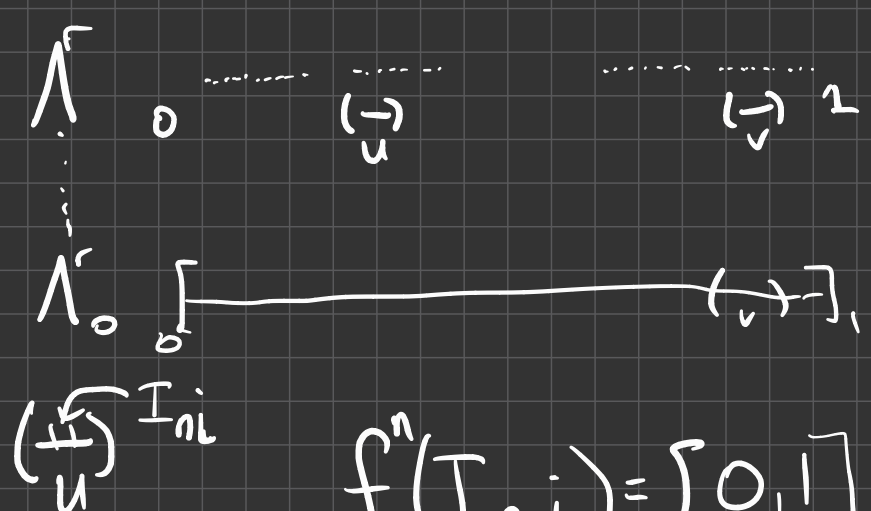
Thm: If $r > 2+\sqrt{5}$, the set $\Lambda^r = \bigcap_{n\geq 0} \Lambda^r_n$ of points which never leave under $f_r(x) = rx(1-x)$ is a Cantor Set

Now we focus on f_r restricted to Λ^r. Endpoints of $I_{n,i}$ are eventually fixed.

Def: A function $f: D \to D$ is continuous is topologically transitive if for any open sets U & V which intersect D, there is a $x \in U \cap D$ and $n \geq 0$ s.t. $f^n(x) \in V$.

Prop: If $r > 2+\sqrt{5}$, then $f_r : \Lambda^r \to \Lambda^r$ is topologically transitive

Proof:



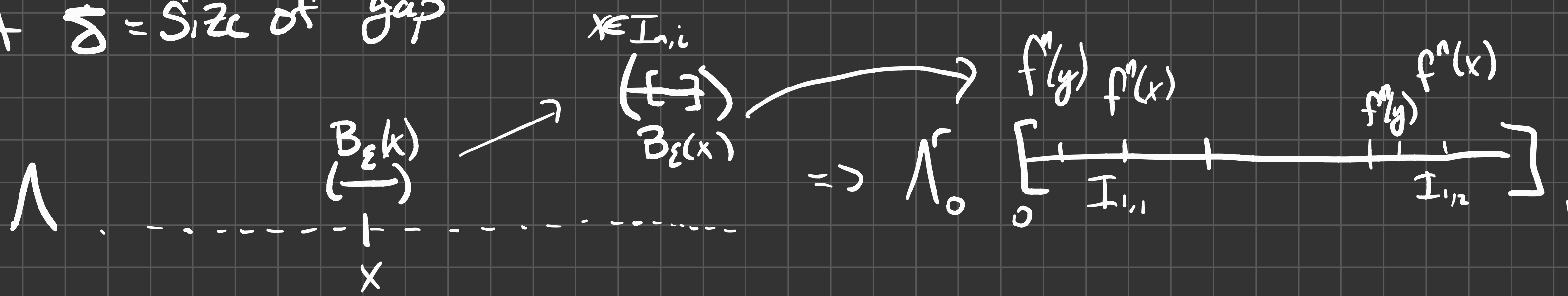
Bijectively, so given $y \in V \cap \Lambda^r$ you can find a $x \in I_{n,i} \subset U$ s.t. $f^n(x) \in V \cap \Lambda^r$

$$f^n(I_{n,i}) = [0,1]$$

Def: $f: D \to D$ exhibits sensitive dependence on initial conditions if there is a $\delta > 0$ s.t. for all $x \in D$ and $\varepsilon > 0$, there is a $x \neq y \in D$ and $n \geq 0$ s.t. $|x-y| < \varepsilon$ and $|f^n(x) - f^n(y)| > \delta$

Prop: $f_r : \Lambda^r \to \Lambda^r$ has sensitive dependence on initial conditions if $r > 2 + \sqrt{5}$

Pf: Let δ = size of gap



Since $f^n(I_{n,i}) = [0,1]$. Suppose $f^n(x)$ is on the left side of the gap. Pick $z \in \Lambda^r \cap$ (Right side of gap)
Since $f^n|_{I_{n,i}}$ is a bijection, there is a $y \in I_{n,i} \cap \Lambda^r \subset B_\varepsilon(x) \cap \Lambda^r$ s.t. $f^n(y) = z$, and $|f^n(y) - f^n(x)| \geq \delta$ (size of gap)

<u>Def (Devaney)</u>: $f: D \to D$ is chaotic if

BAD Definition

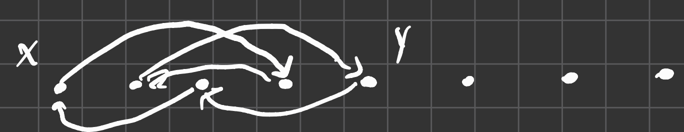
(a) the periodic points are dense in D

(b) f is topologically transitive

(c) f has sensitive dependence on initial dependence

<u>Prop</u>: If $f: D \to D$ is topolgically transitive, then either D is infinite or D is a periodic orbit (D is finite)

<u>Proof</u>: Sps D is finite:

D:


Pick $\varepsilon = \min_{x,y \in D} \frac{|x-y|}{3}$ and so $B_\varepsilon(z) \cap \{z\} = \{z\}$ for $z \in D$

For any x, y, there is an n s.t. $f^n(x) = y$. Also, there is a m s.t. $f^m(y) = x$. So, x is periodic.

By transitivity, there is one periodic orbit

<u>Thm</u>: Let $f: D \to D$ be continuous where $D \subset \mathbb{R}$ is infinite and has a dense set of periodic points and is topologically transitive.
Then, f has sensitive dependence on initial conditions. So f is chaotic.