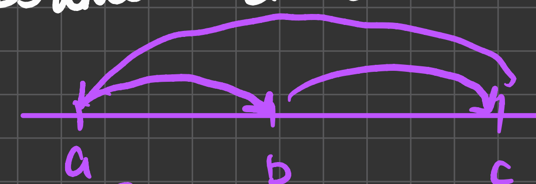
Period 3 Implies Chaos / Sarkovski's Thm

Thm: If f is a continuous function on the real numbers which has a periodic point of prime period 3, then f has a periodic orbit of prime period n for $n \in \mathbb{N}$

Proof: Let a, b, c be the periodic orbit. Assume $a < b < c$ and $f(a) = b$, $f(b) = c$, $f(c) = a$



$[a,b] = I_0$ $\quad$ $[b,c] = I_1$

$[b,c] \subseteq f([a,b])$

$I_0, I_1 \subseteq f([b,c])$

Since $f([a,b]) \supseteq [b,c]$, there is a periodic point w/ prime period 1 in $[b,c]$

Want to show that for all $n \geq 2$, $n \neq 3$, there is a periodic orbit w/ prime period n

Want to find closed intervals $I_1 = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n$ such that:

1) $A_0 = I_1$
2) $f(A_k) = A_{k-1}$, $k = 1, \ldots, n-2$
3) $f^k(A_k) = I_1$, $k = 1, \ldots, n-2$
4) $f^{n-1}(A_{n-1}) = I_0$
5) $f^n(A_n) = I_1$

Claim: Such collection of sets guarantee the existence of the periodic orbit we want

Proof: By (5), $A_n \subseteq f^n(A_n)$ which gives us a fixed point of f^n which is a periodic orbit of period n.

To show its prime period n, take $x \in A_n$ s.t. $f^n(x) = x$.

By (3), $x, f(x), f^2(x), \dots, f^{n-2}(x) \in I_1$

By (4), $f^{n-1}(x) \in I_0$.

If $x = c$, then $f(x) = f(c) = a \notin I_1$

Since $f^{n-1}(x)$ is the only iterate not in I_1, so $1 = n-1$ or $n = 2$. So $x \neq c$

If $x = b$, then $f^2(x) = f^2(b) = a \notin I_1$, so $2 = n-1$ or $n = 3$, but $n \neq 3$, so $x \neq b$.

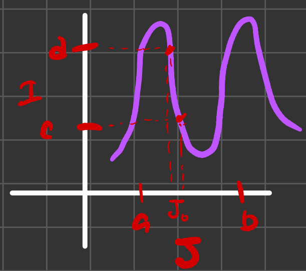
So, $x \in (b, c)$

Since $f^{n-1}(x) \in I_0 = [a, b]$, then $f^{n-1}(x) \neq x$ b/c $I_0 \cap [b, c] = \{\}$. So x is not of prime period $n-1$.

If $f^k(x) = x$ for $k < n-1$, $f^i(x) \in I_1$ for all $i < k$ and x's orbit stays in I_1 which is incompatible w/ (4). So n is the lowest period possible.

Constructions of $\{A_i\}$

Lemma: Let $J=[a,b]$ and $I=[c,d]$. If f is continuous and $f(J)\supseteq I$, then there is a $J_0 \subset J$ s.t. $f(J_0)=I$



Pick $n>1$.

Let $A_0 = I_1$ (1)

Since $f(I_1)\supseteq I$, by Lemma, there is $A_1 \subseteq A_0$ s.t. $f(A_1)=A_0$

Since $A_1 \subseteq A_0$ and $f(A_1)=A_0 \supseteq A_1$, by Lemma, there is $A_2 \subseteq A_1$ s.t. $f(A_2)=A_1$ (2)

and so on for $k=1,2,\dots,n-2$ we get $A_0 \supseteq \dots \supseteq A_{n-2}$

Now, $f^2(A_k) = f(f(A_k)) = f(A_{k-1}) = A_{k-2}$

$f^3(A_k) = f(f(f(A_k))) = f(f(A_{k-1})) = \dots = A_{k-3}$ (3)

$\vdots$

$f^k(A_k) = \dots = f(A_1) = A_0 = I_1$

Since $f^{n-1}(A_{n-2}) = f(f^{n-2}(A_{n-2})) = \dots = f(I_1) \supseteq I_0$. (4)

By Lemma, there is a $A_{n-1} \subset A_{n-2}$ s.t. $f^{n-1}(A_{n-1}) = I_0$

Finally, $f^n(A_{n-1}) = f(f^{n-1}(A_{n-1})) = f(I_0) = I_1$.

(5)

By Lemma, there is a $A_n \subseteq A_{n-1}$ s.t. $f(A_n) = A_{n-1}$ and $f^n(A_n) = I_1$

Q.E.D.

This is a very 1 dimensional phenomenon. In higher dimensions, period 3 orbits do not imply anything.

Def: Sarkovski's ordering of $\mathbb{N}$

$$3 > 5 > 7 > 9 > \dots > 2(3) > 2(5) > 2(7) > 2(9) > \dots > 2^2(3) > 2^2(5) > \dots > 2^n(3) > 2^n(5) > 2^n(5) > \dots$$
$$> 2^n > \dots > 2^2 > 2 > 1$$

Sarkovski's Theorem

If f is continuous and there is a periodic orbit of prime period n, then if $n > m$, there is a periodic orbit of period m.