Periodic Orbits & Stable Sets

Def: If f is a function, then p is a fixed point if $f(p)=p$

Theorem: If $f:[a,b]\to[a,b]$ is continuous, then f has a fixed point

Pf: If $f(a)=a$ or $f(b)=b$, then we're done. Assume $f(a)\neq a$, $f(b)\neq b$.
So, $f(a)>a$ & $f(b)<b$.
Let $g(x)=f(x)-x$ which is continuous. We have $g(a)=f(a)-a>0$ and $g(b)=f(b)-b<0$. By the Intermediate Value Theorem, there is a $c\in(a,b)$ such that $g(c)=0$, so $g(c)=f(c)-c=0$, so $f(c)=c$

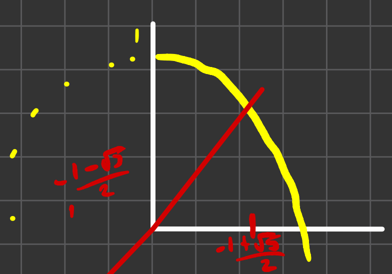
Use I.V.T b/c f is continuous

Must cross $f(x)=x$ by I.V.T

Ex: $f(x)=1-x^2$ has a fixed point

$$f(x)=1-x^2=x$$

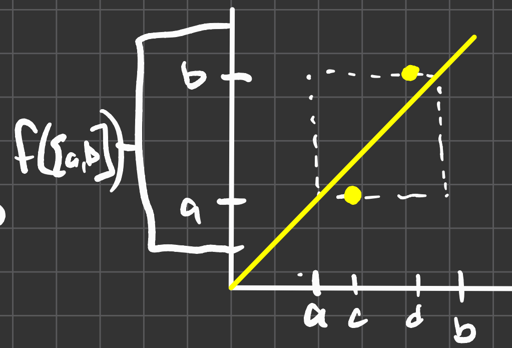
$$x^2+x-1=0 \Rightarrow x=\frac{-1\pm\sqrt{5}}{2}$$



Theorem: Let $f:[a,b] \to \mathbb{R}$ be continuous such that $[a,b] \subseteq f([a,b])$.
Then, f has a fixed point

Proof: If $f(a)=a$ or $f(b)=b$, done.
Assume $f(a) \neq a$ & $f(b) \neq b$.

There is a $c>a$ such that $f(c)=a$
and $d<b$ such that $f(d)=b$



Let $g(x)=f(x)-x$ so $g(c)=f(c)-c=a-c<0$
and $g(d)=f(d)-d=b-d>0$.
By I.V.T, there is a zero of g between c & d

Functions w/ no fixed points:

$f(x)=x+2$ b/c $f([0,1])=[2,3]$

$[a,b] \subseteq f([a,b])$ is necessary for fixed points

Def: If f is a function, then p is a periodic point w/ period n if
$f^n(p)=f \circ f \circ \dots \circ f(p)=p$. p has prime period n if $f^k(p) \neq p$ for all $0<k<n$

The set of points $\{x, f(x), f^2(x), \dots\}$ is the orbit of x. If $\underline{x}$ is periodic, this is called a periodic orbit or cycle

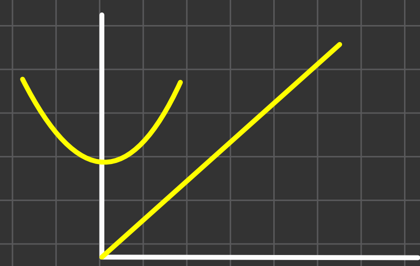
Ex: $f(x) = -x^5$

- Fixed points: 0
- periodic orbits of period 2: $\{0, 1, -1\}$
- Periodic Orbits of prime period 2: $\{1, -1\}$

Ex: $f(x) = x^2+1$

- Fixed points: $x^2+1 = x \Rightarrow x^2 - x + 1 = 0 \Rightarrow \frac{-1 \pm \sqrt{-3}}{2}$ not real

  ↳ No fixed points



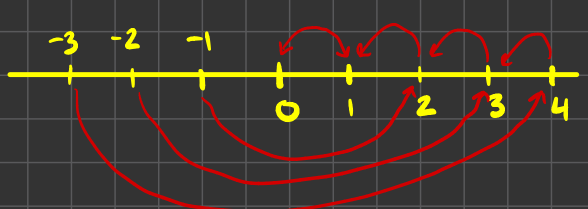
Def: x is eventually fixed if there is an N such that if $k > N$

$$f^{k+n}(x) = f^k(x) \text{ for all } n > 0$$

Def: x is eventually periodic w/ period k if there is N such that $f^{n+k}(x) = f^n(x)$ for any $n > N$

Ex: $h: \mathbb{Z} \to \mathbb{Z}$

$h(x) = |x-1|$, $h(0) = 1$, $h(1) = 0$



All points other than 0, 1 are eventually periodic

Ex: $h(x) = rx(1-x)$ ← Logistic

Fixed point: 0

Eventually Fixed: $1 \to h(1) = 0$

Def: If f is a function and p a periodic point of period k, then x is forward asymptotic to p if $\lim_{n\to\infty} f^{kn}(x) = p$.

$x, f^{k}(x), f^{2k}(x), f^{3k}(x), \ldots \to p$

Def: The stable set of p $W^s(p)$ is the set of all points which are forward asymptotic to p.

Def If $|x|, |f(x)|, |f^2(x)|, \ldots \to \infty$, then x is forward asymptotic to ∞

x belongs to $W^s(\infty)$

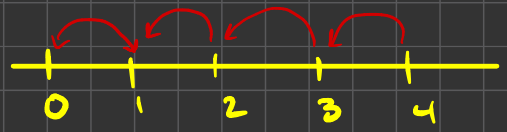
Ex: $f(x) = \sqrt{x}$

0 is fixed $\to W^s(0) = 0$

1 is fixed $\to W^s(1) = (0, \infty)$

Ex: $h: \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$

$h(x) = |x-1|$



$W^s(1)$ = odd natural #s

$W^s(0)$ = even natural #s

Theorem: If $p_1 \neq p_2$ & are periodic, then

$$W^s(p_1) \cap W^s(p_2) = \emptyset$$

Proof: $\lim_{n \to \infty} f^{nk_i}(x) = p_i$ for $x \in W^s(p_i)$, k_i period of p_i

Then, if $x \in W^s(p_i) \cap W^s(p_2)$,

$$\lim_{n \to \infty} f^{nk_1k_2}(x) = p_1 \neq \lim_{n \to \infty} f^{nk_1k_2}(x) = p_2$$

Plotting Orbits