Symbolic Dynamics

Def: The set of all infinite sequences or strings of zeroes & ones is called the symbol space of 0,1 & is denoted by Σ_2
(Book)

$\Sigma_2 = \{0,1\}^{\mathbb{N}}$

(In Practice) $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$, $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$

↪ bi-infinite strings

$\bar{x} = (x_1, x_2, x_3, \dots) \in \Sigma_2^+$

$\bar{x} = (\dots, x_{-1}, x_0, x_1, \dots) \in \Sigma_2$

Defn: For $\bar{x}, \bar{y} \in \Sigma_2^+$, let $\overset{\text{Distance}}{d(x,y)} = \sum_{i>0} \frac{|x-y|}{2^i}$
(Book)

(Alternative): $\bar{d}(\bar{x},\bar{y}) = 2^{-K(\bar{x},\bar{y})}$, $K(\bar{x},\bar{y})$ = smallest idx where $x_i \neq y_i$

Def: Let X be a set. If there is a fctn $d: X \times X \longrightarrow \mathbb{R}$ s.t.

1) $d(x,y) \geq 0$; $d(x,y) = 0$ iff $x = y$ Positive Distances

2) $d(x,y) = d(y,x)$ Symmetry

3) $d(x,y) + d(y,z) \geq d(x,z)$ Triangle Inequality

Then, d is a metric on X and X is a metric space

Def: Let d be a metric on set X.

a) $U \subset X$ is an open set if for any $x \in U$, there is an $\varepsilon > 0$ s.t. if $d(x,y) < \varepsilon$, then $y \in U$

b) For $x \in X$ and $\varepsilon > 0$, the set $N_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$ is the ε-neighborhood. $B_\varepsilon(x) = \varepsilon$-ball around x

c) A sequence of pts $x_1, x_2, x_3, \ldots$ converges to x^* if for any ε, there is an idx N s.t. $d(x^*, x_i) < \varepsilon$ if $i > N$

d) x is an accumulation/limit pt of X if for any $\varepsilon>0$, there is a $x \neq y \in X$ s.t. $y \in N_\varepsilon(x)$

e) A set V is closed if it contains all of its limit pts.

f) A set $A \subset X$ is dense in X if for any $x \in X \setminus A$ and $\varepsilon>0$, there is a $y \in A$ s.t. $y \in N_\varepsilon(x)$

g) If d' is a metric on Y, then $f: X \to Y$ is continuous if for any converging sequence $x_1, x_2 \to x^*$, the sequence $f(x_1), f(x_2), \ldots \xrightarrow[n]{} f(x^*)$ in Y

Properties:

1) A set U is open if for any $x \in U$, there is a $\varepsilon>0$ s.t. $B_\varepsilon(x) \subset U$

2) $B_\varepsilon(x) = N_\varepsilon(x)$ is open

3) The following are equivalent:

a) x is an accumulation pt

b) $\exists$ a sequence x_1, x_2 converging to x w/ $x_i \neq x$ for all i.

4) The compliment of an open set is closed.

The compliment of a closed set is open

5) A fctn $f: X \to Y$ is continuous iff $f^{-1}(U)$ is open wherever $U \subset Y$ is open

Recall: $\bar{d}(\bar{x}, \bar{y}) = 2^{-K(\bar{x}, \bar{y})}$

Given $\bar{x} = (x_1, x_2, \ldots) \in \Sigma_2^+$ what is $N_{1/2}(\bar{x})$

$$N_{1/2}(\bar{x}) = \{\bar{y} \in \Sigma_2^+ : y_1 = x_1\} = \{x_1, *, *, *\}$$

$$N_{2^{-n}}(\bar{x}) = \{\bar{y} \in X : y_i = x_i \text{ for } i \leq n\}$$

What are the accumulation pts of $N_{2^{-n}}(x)$?

Claim: $N_{2^{-n}}(\bar{X})$ consists of all accumulation pts of $N_{2^{-n}}(\bar{X})$.

Proof: Pick $\bar{\omega} \in N_{2^{-n}}(\bar{X})$:

$$\bar{\omega} = (X_1, X_2, \dots, X_n, \omega_{n+1}, \omega_{n+2}, \dots)$$

$$\bar{\omega}_1 = (X_1, X_2, \dots, X_n, \omega_1, \dots, \underset{\substack{\uparrow \\ \text{flip}}}{\omega_k'}, \omega_{k+1}, \dots)$$

Doing this $\bar{\omega}_k \to \bar{\omega}$ so $\bar{\omega}$ is a limit pt

$V_k = (X_1, \dots, X_n', \text{vary tail})$ cannot limit to $N_{2^{-n}}(\bar{X})$

Conclusion: $N_2(\bar{X}) \subset \Sigma_2^+$ are both closed & open, clopen?

Consider $T_0 = \{ \bar{X} \in \Sigma_2^+ : X_i = 0$ for all i large enough $\}$

<u>Claim</u>: $T_0 \subset \Sigma_2^+$ is dense

<u>Pf</u>: Pick $\bar{y} \in \Sigma_2^+$. Let $y_n = (y_1, y_2, \dots, y_n, \bar{0})$

so $y_n \in T_0$ and $y_n \xrightarrow[n]{} y$

Ultrametric / Non-Archimedean Spaces:

<u>Def</u>: The <u>shift map</u> $\sigma : \Sigma_2^+ \to \Sigma_2^+$ is the function defined by shifting to the left / truncating the first coordinate.

$\bar{X} \in \Sigma_+^2$, $\bar{X} = X_1 X_2 X_3 \dots$

$\sigma(\bar{X})_i = X_{i+1}$ $\qquad$ $\sigma(\bar{X}) = X_2 X_3 X_4 \dots$

There is also a shift map $\sigma : \Sigma_2 \to \Sigma_2$ which shifts to the left.

$\sigma(\ldots x_{-1} x_0 . x_1 \ldots) = \ldots x_{-1} x_0 x_1 . x_2 \ldots$

<u>Proposition</u>: The shift map is continuous

<u>Pf</u>: Pick $\bar{x} \in \Sigma_2^+$. $\bar{x} = x_1 x_2 \ldots$

Pick $\bar{x} \neq \bar{y}_k = x_1 x_2 \ldots x_k \bar{0}$

$\sigma(\bar{y}_k) = x_2 x_3 \ldots x_k \bar{0}$ $\quad\quad$ $\sigma(\bar{x}) = x_2 x_3 \ldots x_k x_{k+1} \ldots$

if $\bar{y}_k \xrightarrow{k} \bar{x}$ then $\sigma(\bar{y}_k) \xrightarrow{k} \sigma(\bar{x})$

<u>Proposition</u>: The shift map has the following properties:

(1) The set of periodic pts is dense in Σ_2^+

(2) The set of periodic pts of period n is 2^n elements

(3) The set of eventually periodic pts which are not periodic is dense in Σ_2^+

(4) There is an element w/ a dense orbit.

(5) The set of pts which are neither periodic or eventually periodic is dense in Σ_2^+

Pf:

1) Pick arbitrary $\bar{x} \in \Sigma_2^+$ and $\varepsilon = 2^{-n}$.
Let $\bar{y} = x_1 x_2 \ldots x_n x_{n+1}\, x_1\, x_2 \ldots$
Then $\bar{y}$ is periodic & $d(\bar{x}, \bar{y}) \leq \varepsilon$

2) Let L_n be the set of strings of length n.
Then, $|L_n| = 2^n$. Each $w \in L_n$ gives the periodic orbit
$x_w = \bar{w} \in \Sigma_2^+$ of period n.

3) Pick $\bar{x} \in \Sigma_2^+$ and $\varepsilon = 2^{-n}$.
Let $\bar{y} = x_1 x_2 \ldots x_{n+1}\bar{0}$. Then $\bar{y}$ is eventually periodic &
$d(\bar{x}, \bar{y}) \leq \varepsilon$ (assuming $x_1 x_2 \ldots x_{n+1} \neq \bar{0}$)

4) Let $\bar{x} = \underbrace{01}_{\substack{\text{all string}\\ \text{of length}\\ 1}}\underbrace{00011011}_{\substack{\text{all strings of length}\\ 2}}\underbrace{000\,001\,010\,011}_{\text{all strings of length 3}}$

If you want $\bar{x}$ to visit an open set determined by a string of length K.

Then, after roughly 2^{K+1} iterates, $\sigma^i(\bar{x})$ will visit that open set. Since all open sets are constructed from determining the first J symbols,

$J^i(\bar{x})$ will eventually enter the open set.

5) The orbit from 4 is a subset of this set & it's already dense

Def: Let $f: D \to D$ where D is a subset of a metric space. f is topologically transitive if for any open U, V w/ non-empty intersection w/ D, there is a $n > 0$ such that $f^n(U \cap D) \cap (V \cap D) = \emptyset$
or there is $z \in U \cap D$ s.t. $f^n(z) \in V$

Proposition: Let $f: D \to D$ be a fctn on a subset of a metric space.
If the periodic pts are dense & there is a pt w/ a dense orbit, then f is topologically transitive.

Def: Let $f: D \to D$ be defined on a subset of a metric space.
Then, f has sensitive dependence on initial conditions if there is a $\delta > 0$ s.t. for any $x \in D$ & $\varepsilon > 0$, $y \in N_\varepsilon(x)$ there is a $n > 0$ s.t. $d(f^n(x), f^n(y)) > \delta$

Def: A fctn $f: D \to D$ is Chaotic if:

1) The periodic points are dense
2) f is topologically transitive
3) f has sensitive dependence on initial conditions

Thm: Let D be an infinite subset of a metric space and $f: D \to D$ is continuous.
Then, (1) & (2) $\Rightarrow$ (3)

Prop: $\sigma: \Sigma_2^+ \to \Sigma_2^+$ is topologically transitive

Pf: Let U, V be determined by strings w_u, w_v:
$U = \{\bar{x} \in \Sigma_2^+ : \bar{x} = w_u *** \ldots\}$ $\quad$ $V = \{\bar{x} \in \Sigma_2^+ : w_v *** \ldots\}$

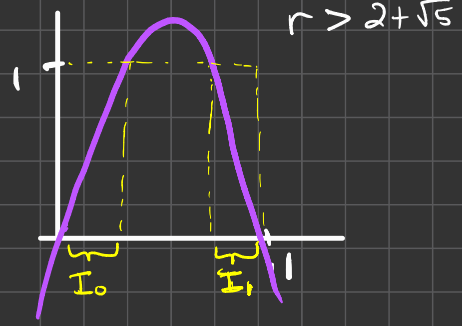
Pick $\bar{x} = w_u w_v ** \ldots \in U$
Then, $\sigma^{|w_u|}(\bar{x}) = w_v *** \ldots \in V$

Corollary: The shift map is Chaotic

Back to Logistical Map

$h_r(x) = rx(1-x)$

Defined Λ^r to be the set of pts in $[0,1]$ which never leave $[0,1]$.



We want to find $\tau : \Lambda^r \to \Sigma_2^+$ s.t. $\tau \circ h_r = \sigma \circ \tau$.

Given $x \in \Lambda^r$, we want to describe its itinerary/address in that

$$\tau_r(x)_i = \begin{cases} 0, & \text{if } h_r^{i-1}(x) \in I_0 \\ 1, & \text{if } h_r^{i-1}(x) \in I_1 \end{cases}$$

We do this & it satisfies $\boxed{\tau_r \circ h_r = \sigma \circ \tau_r}$

Need to check if τ_r is a homeomorphism

1) Bijective
2) Continuity
3) Inverse Continuity

$\Sigma_n^+ = \{0, 1, \dots, n-1\}^{\mathbb{N}}$ = n-shift

$\sigma : \Sigma_n^+ \to \Sigma_n^+$ shift map, continuous

Def: $X \subset \Sigma_n^+$ is a subshift if

1) $\sigma(X) = X$

2) X is closed

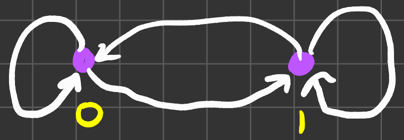
Ex: $X \subset \Sigma_2^+$ = sequences w/ no repeated 1s.

↳ subshift of Σ_2^+ $\longrightarrow$ $B = \{11\}$

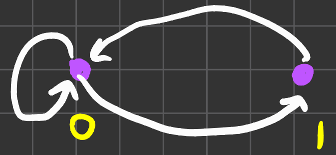
It is a subshift of finite type (SFT) if there is a finite list B of forbidden strings which do not appear in elements of X

Graph Representations

Any finite directed graph $\Gamma = (V, E)$ defines a vertex shift which is a SFT in $\Sigma_{\#V}^+$ by $\Sigma_\Gamma^V = \{ \Sigma_{\#V}^+ :$ there is an infinite path in Γ visiting V_i at step i $\}$



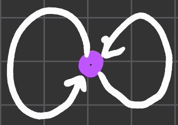
$\Rightarrow \Sigma_\Gamma^v = \Sigma_2^+$



$\Rightarrow$ This gives example from Thurs as its vertex subshift. (No repeating 1s)

Any finite directed graph Γ defines an edge shift, an SFT, in $\Sigma^+_{\#E}$ by:

$$\Sigma_\Gamma^e = \left\{ \bar{x} \in \Sigma^+_{\#E} : \begin{array}{l}\text{there is an infinite path}\\ \text{in } \Gamma \text{ visiting } e_i \text{ at step } i\end{array} \right\}$$



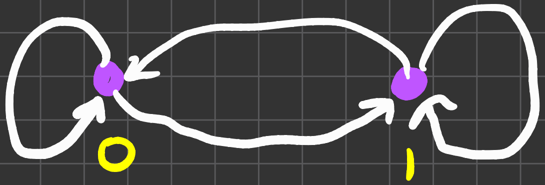
$\Rightarrow$ Edge shift is Σ_2^+

Fact: Every SFT has a graph representation as an edge shift or a vertex shift. The graph depends on list B of forbidden words.

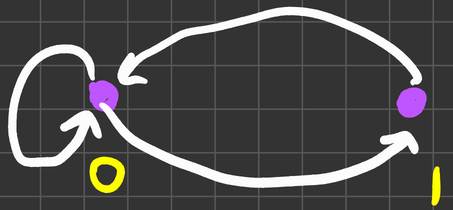
If X is given by a vertex shift, there is an associated adjacency matrix.

$$(A_\Gamma^v)_{ij} = \begin{cases} 1 & \text{if there is edge btwn } i,j \\ 0 & \text{else} \end{cases}$$

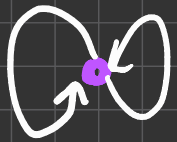
If X is given by an edge shift, then there is an associated adjacency matrix: $(A^e_\Gamma)_{ij}$ = # of edges btwn v_i, v_j



$\longrightarrow (A^v_\Gamma) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to$ eigenvalue: 2



$\longrightarrow (A^e_\Gamma) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \to$ eigenvalue: $\frac{1+\sqrt{5}}{2}$

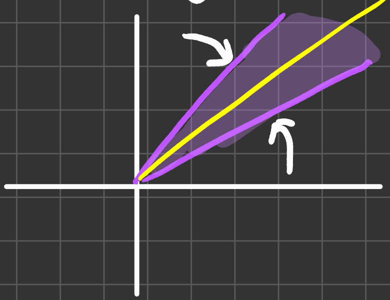


$\longrightarrow (A^e_\Gamma) = (2) \to$ eigenvalue: 2

Def: A non-negative square matrix is primitive if there is an $n > 0$ s.t. A^n is positive.

Thm (Perron-Frobenius): Let A be primitive.

1) A has a positive eigenvalue, λ_{pf}
2) λ_{pf} is a simple root of char poly of A
3) λ_{pf} has a positive eigenvector $\bar{v}_{pf}$
4) λ_{pf} is the largest eigenvalue in norm
5) Any non-neg eigenvalue is a multiple of $\bar{v}_{pf}$



$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2$$

Stretches w/in cone and eventually approaches the 1 dimensional line.

Back to Shift of Finite Type:

Question: How many pts of period n does $\sigma : X \to X$?

Take (A_Γ^n) and add all diagonals $\Rightarrow$ $\text{trace}(A_\Gamma^n)$

This is the size of strings of length n that you see $|W_n|$

Def: The spectral radius, ρ, of A is the modulus/norm of the largest eigenvalue

Ex: If A is primitive: $\rho_A = \lambda_{pf}$

Def: The topological entropy of $\sigma : X \to X$ is:

$$h(\sigma) = \lim_{n\to\infty} \frac{\log|W_n|}{n}$$

→ log base doesnt matter

→ exponential growth rate of periodic orbits

For 2-shift: $h(\sigma) = \lim_{n\to\infty} \frac{\ln 2^n}{n} = \ln 2$

Thm: If $\sigma : X \to X$ is a SFT w/ primitive matrix A,

then $\boxed{h(\sigma) = \ln(\rho_A) = \ln(\lambda_{pf})}$

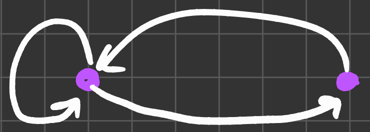
Proof: $|W_n| = \text{trace}(A^n) =$ sum of eigenvalues of A^n

$$= \lambda_{pf}^n + \dots$$

Since λ_{pf} is largest by norm, there is constant $C > 1$ such that:

$$\frac{1}{C}\lambda_{pf}^n \le |W_n| \le C\lambda_{pf}^n$$

So, $$\underset{\substack{\| \\ \log \lambda_{pf}}}{\lim_{n\to\infty} \frac{\log(\frac{1}{C}\lambda_{pf}^n)}{n}} \le \lim_{n\to\infty} \frac{\log|W_n|}{n} \le \underset{\substack{\| \\ \log \lambda_{pf}}}{\lim_{n\to\infty} C\lambda_{pf}^n}$$



$$h(\sigma) = \ln\left(\frac{1+\sqrt{5}}{2}\right)$$

# of periodic points of period n is

$\approx \lambda_{pf}^n \approx F_n$ (nth Fibonacci #)

$h(\sigma)$ measures the max uncertainty in exponential form that you have in extending a word in the subshifts to a larger word.

if $h(\sigma)=0$, it does not mean you have no uncertainty, it means it's not exponential.

↰ Not Actual Def of Topological Entropy. But for SFTs, they are equivalent

$$\begin{tikzcd}
X \arrow[r, "\sigma"] \arrow[d, "\tau"'] & X \arrow[d, "\tau"] \\
Y \arrow[r, "f"] & Y
\end{tikzcd}$$

is a topological conjugacy, then $h(f) = h(\sigma)$

<u>Corollary</u>: For $r > 2+\sqrt{5}$, the logistic map $h_r : \Lambda^r \to \Lambda^r$ has topological entropy $\ln(2)$

Positive Topological Entropy is a better criterion for Chaos