Topological Conjugacy

Not on Exam 2

Def: Let $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ be some functions.
Then, f_1 & f_2 are topologically conjugate if there is a homeomorphism $h : X_1 \to X_2$ such that $h \circ f_1 = f_2 \circ h$

$$h \circ f_1 = f_2 \circ h \xrightarrow[\text{by } h^{-1} \text{ on right}]{\text{compose}} h \circ f_1 \circ h^{-1} = f_2 \xrightarrow[\text{by left}]{\text{compose}} f_1 = h^{-1} \circ f_2 \circ h$$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[ddd, bend right] & f_1(x) \arrow[ddd, bend left] \\
X_1 \arrow[r, "f_1"] \arrow[d, "h"'] & X_1 \arrow[d, "h"] \\
X_2 \arrow[r, "f_2"] & X_2 \\
h(x) \arrow[r, mapsto] & f_2(h(x)) = h(f_1(x))
\end{tikzcd}
\qquad
\begin{tikzcd}
h^{-1}(x) \arrow[r, mapsto] & f_1(h^{-1}(x)) = h^{-1}(f_2(x)) \\
X_1 \arrow[r, "f_1"] & X_1 \\
X_2 \arrow[r, "f_2"] \arrow[u, "h^{-1}"] & X_2 \arrow[u, "h^{-1}"'] \\
x \arrow[r, mapsto] \arrow[uuu, bend left] & f_2(x) \arrow[uuu, mapsto, bend right]
\end{tikzcd}$$

Note: Linear Algebra Diagonalization/Change of Basis is topological conjugacy.

Prop: Let f_1, f_2 be topologically conjugate by a homeomorphism $h: X_1 \to X_2$
Then:

(1) $U \subset X_1$ is open if & only if $f_1(U)$ is open

(2) The sequence $x_1, x_2, x_3, \ldots,$ converges to $x \in X$
iff $h(x_1), h(x_2), \ldots$ converges to $h(x)$

(3) A set $C \subset X_1$ is closed iff $h(C)$ is closed in X_2

(4) A set D is dense in X_1 iff $h(D)$ is dense in X_2

Thm: Let f_1, f_2 be topologically conjugated by h. Then,

(1) h^{-1} is a topological conjugacy between f_1, f_2

(2) $h \circ f_1^n = f_2^n \circ h$ for all $n \geq 0$

(3) $p \in X_1$ is periodic w/ prime period k
iff $h(p)$ is f_2-periodic w/ prime period k

(4) If $p \in X_1$ is f_1-periodic, then $W^s(p)$ satisfies

$$W^s(h(p)) = h(W^s(p))$$

(5) The f_1-periodic points are dense in X_1 iff the f_2-periodic points are dense in X_2

(6) f_1 is topologically transitive iff f_2 is topologically transitive

(7) If $X_1, X_2 \subseteq \mathbb{R}$ & f_1, f_2 are continuous, then f_1 is chaotic iff f_2 is chaotic

Sketch of Proofs:

(1) Done. See above.

(2) $h \circ f_1 = f_2 \circ h \rightarrow h \circ f_1^2 = h \circ f_1 \circ f_1 = f_2 \circ h \circ f_1$
$$= f_2 \circ f_2 \circ h = f_2^2 \circ h$$

⋮

Induction ... Done

(3) If $f_1^k(p) = p$, then $h \circ f_1^k(p) = h(p)$

$h \circ f_1^k(p) = f_2^k(h(p)) \xrightarrow{\downarrow}$ so $h(p)$ is periodic w/ period k

If k is the prime period for p, if $n<k$ satisfies

$$f_2^n(h(p)) = h(p) \overset{h^{-1}}{\Longrightarrow} f_2^n(p) = p$$

so prime period is preserved.

(4) If $x \in W^s(p)$ then $f^{kn}(x) \xrightarrow{n} p$.

Since h preserves converging sequences,

$$h(f_1^{kn}(x)) = f_2^{kn}(h(x)) \longrightarrow h(p)$$

so $h(x) \in W^s_{f_2}(h(p))$

(5) By 3 and the fact that h preserves dense sets.

(6) Assume f_1 is topologically transitive. Want to show f_2 is topologically transitive.

Pick open sets $U_2, V_2 \subset X_2$

Let $V_1 = h^{-1}(V_2)$, $U_1 = h^{-1}(U_2)$, which are open since f_1 is topologically transitive.

There is an $x \in U_1$, $n \geq 0$ s.t. $f_1^n(x) \in V_1$. By the Conjugacy,

we have $h(x) \in U_2$ and $f_2^n(h(x)) = h \circ f_1^n(x) \in V_2$

so, f_2 is topologically transitive

(7) Under the hypothesis, we get sensitive dependence on initial conditions

<u>Ex</u>: $f_1 : (0,1) \to (0,1)$ $\quad f_1(x) = x^2$ ← No Sensitive Dependence (Contracts)

$f_2 : (1,\infty) \to (1,\infty)$ $\quad f_2(x) = x^2$ $\quad h(x) = \frac{1}{x}$

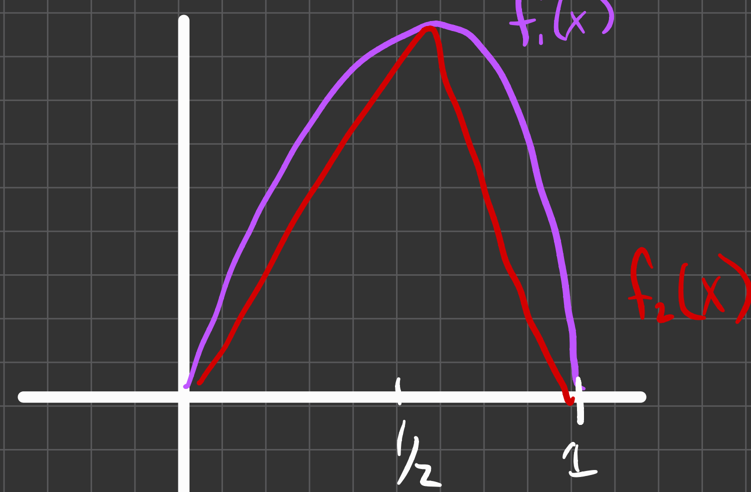
Sensitive Dependence on Initial Conditions

h is a topological conjugacy between f_1, f_2

<u>Thm</u>: Let $f_1(x) = 4x(1-x)$ on $[0,1]$ and $f_2(x) = \begin{cases} 2x, & [0,\frac{1}{2}] \\ 2-2x, & [\frac{1}{2},1] \end{cases}$

Then, f_1 & f_2 are topologically conjugated by $h(x) = \sin^2(\frac{\pi}{2}x)$

<u>Corollary</u>: f_1 is Chaotic



<u>Proof</u>: Need to show $h(x)$ is injective, surjective, has cont inverse, and satisfies $h \circ f_1 = f_2 \circ h$