Optimization

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Abstract

History of Cylinders and The 55-gallon Tight Head Steel Drum

Circular cylinders have been used for hundreds of years as storage for transportation. Originally made out of wood, these cylinders held solid and liquid goods, such as oil. In 1900, the world had an increasing demand for oil, and drilling was taking place all over the world. But these wooden barrels were not a good mode of transportation for oil. There was a need for manageable-sized, durable, and leak-proof barrels.

Elizabeth J. Cochran Seaman, A.K.A Nellie Bly, manufactured the first 55-gallon steel drum and patented her design in 1905. This design is actually the same model that we use today.

The use of these 55-gallon steel drums became widespread during World War II, where The Navy used these drums to store gasoline for its air crafts.

But, There Is a Problem...

Though Bly's steel drum design was amazing and innovative for its time, it is actually not the most optimal cost effective design. This report will show that Bly's steel drum design does not meet the optimal requirements for reducing the manufacturing cost.

How Do We Solve this Problem? With Optimization

Optimization is the process of finding the maximum or minimum value of a function for some constraint, which has to be true regardless of the solution. In other words, we will be using Optimization to maximize the cost efficiency of the steel drum.

To conduct this kind of Optimization problem, we must first recognize that there are two unknowns. So, we will end up creating two equations with two unknowns, one will be a constraint (in this case, cost) function and one will be a volume function. We will write a cost equation and write it in terms of a single variable. Then we will plug in that variable into the volume equation, take the derivative of that volume equation to find critical points, and finally plug in the critical values into the second derivative to determine if it is a maximum, minimum , or neither.

The Project

As well as proving that the dimensions of the 55-gallon Tight Head Steel Drum are not the best dimensions to minimize costs, a simpler cylinder optimization problem will be explained and solved to get the reader comfortable and familiarized with the optimization process.

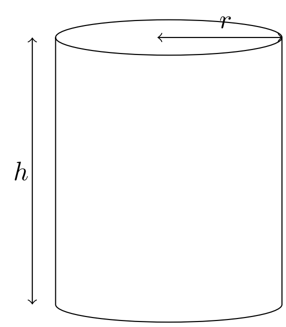
Context/Work

Initial Cylinder Optimization Problem

To get comfortable with optimization, we were asked to solve the following cylinder optimization problem:

> A closed right circular cylinder (i.e. top and bottom included) has a surface area of 100 square centimeters. What should the radius and height be in order to provide the largest possible volume? Find the result if the surface area is S square centimeters.

Below is a diagram to help you visualize the problem:



To begin, we will be using the surface area and volume equations shown below to solve for the height and radius that will provide the largest possible volume:

$$SA = 2(\pi r^2) + 2\pi r h = 100$$
$$V = \pi r^2 h$$

In order to continue, we must solve the surface area equation for one variable. In this case, I chose to solve for h:

$$2(\pi r^2) + 2\pi rh = 100$$
$$2(\pi r^2) - 100 = -2\pi rh$$
$$\frac{2(\pi r^2) - 100}{-2\pi r} = h$$
$$\frac{2(\pi r^2)}{-2\pi r} - \frac{100}{-2\pi r} = h$$
$$\frac{100}{2\pi r} - r = h$$
$$h = \frac{50}{\pi r} - r$$
$$V = 50r - 3\pi r^2$$

Now, we plug h into the volume equation, giving us volume in terms of r only:

$$V = \pi r^2 h$$
$$V = \pi r^2 (\frac{50}{\pi r} - r)$$
$$V = 50r - 3\pi r^3$$

Now, we must take the first derivative, $\frac{dV}{dr}$, to find the critical points, which will then allow us to determine the maximum/minimum height and radius after taking the second derivative, $\frac{d^2V}{dr^2}$:

$$\frac{dV}{dr} = 50 - 3\pi r^2$$
$$0 = 50 - 3\pi r^2$$
$$3\pi r^2 = 50$$
$$r^2 = \frac{50}{3\pi}$$
$$r = \pm\sqrt{\frac{50}{3\pi}}$$

We will only use the positive value of r, since a radius cannot be negative:

$$r = \sqrt{\frac{50}{3\pi}}$$

Now, we can take the second derivative of the volume equation to confirm if it has a maximum or minimum:

$$\frac{d^2V}{dr^2} = -6\pi r$$

If we plug in r for $\frac{d^2V}{dr^2}$, it will produce a negative value, confirming that the function is concave down and therefore has a maximum volume at $r = \sqrt{\frac{50}{3\pi}}$.

Now that we have found r, we can plug it into the surface area equation to obtain the optimal height:

$$SA = 2(\pi r^2) + 2\pi rh = 100$$

$$2(\pi r^2) + 2\pi rh = 100$$

$$2\pi(\sqrt{\frac{50}{3\pi}})^2 + 2\pi(\sqrt{\frac{50}{3\pi}})h = 100$$

$$2\pi \cdot \frac{50}{3\pi} - 100 = -2\pi(\sqrt{\frac{50}{3\pi}})h$$

$$\frac{100}{3} - 100 = -2\pi(\sqrt{\frac{50}{3\pi}})h$$

$$h = \frac{\frac{100}{3} - 100}{-2\pi(\sqrt{\frac{50}{3\pi}})}$$

So, the optimal dimensions for this cylinder are:

- radius: $\sqrt{\frac{50}{3\pi}}$
- height: $\frac{\frac{100}{3}-100}{-2\pi(\sqrt{\frac{50}{3\pi}})}$

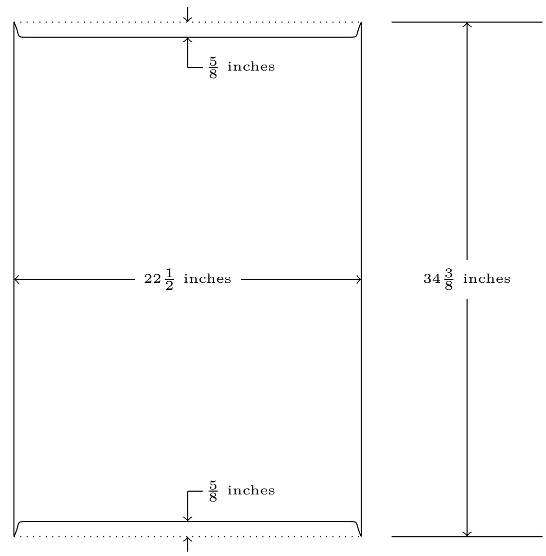
To find the same result if the surface area is S centimeters, we must remember that S and 100 are the same element of the equation and are interchangeable if we replace 100 with S using the un-simplified (meaning that you don't divide by 2 and reduce 100 to 50) versions of the two equations, or $\frac{\frac{100}{3}-100}{-2\pi(\sqrt{\frac{100}{6\pi}})}$ and $\sqrt{\frac{100}{6\pi}}$.

So, the optimal dimensions for this cylinder with a surface area of S are:

- radius: $\sqrt{\frac{S}{6\pi}}$
- height: $\frac{\frac{S}{3}-S}{-2\pi(\sqrt{\frac{S}{6\pi}})}$

The 55-gallon Tight Head Steel Drum Problem

The specification diagram from The American National Standards Institute (ANSI) document is given below:



Cost and Material Parameters and Specifications to Account For:

- 18 gauge steel is 45 cents per square foot ($\frac{45}{144}$ cents per square inch)
- 20 gauge steel is 45 cents per square foot ($\frac{34}{144}$ cents per square inch)
- Welding and pressing/sealing cost is 10 cents per foot ($\frac{10}{12}$ cents per inch)
- Cutting steel costs 2 cents per foot ($\frac{2}{12}$ cetns per inch)

Other Parameters and Specifications:

- The vertical seam on the cylinder is welded together
- The top and bottom are attached by a pressing/sealing machine. The pressing/sealing process requires approximately $\frac{13}{16}$ inches from the cylinder and $\frac{3}{4}$ inches from the disk to be curled together. Hence, these inches are lost in the final dimensions
- The top and bottom are set down $\frac{5}{8}$ inches into the cylinder.

Work:

Some equations and variables must first be defined before solving this problem. In this problem, h will be the height that contributes to the volume of the drum and r will be the radius of the circle that contributes to the volume of the drum.

- Area of the circle (top/bottom) that contributes to the volume: πr^2
- Area of the circle that must be used: $\pi(r + \frac{5}{8})^2$
- Area of the sheet that must be bought per circle: $[2(r + \frac{5}{8})]^2$
- Area of the cylinder (using the rectangular sheet): $2\pi r \cdot A = (h + \frac{13}{8})$
- Vertical seam of the cylinder: $h + \frac{13}{8}$
- Circumference of each circle: $2\pi(r + \frac{5}{8})$
- Volume of the drum:

$$V = 55 \text{ gallons} \cdot \tfrac{231 in^3}{1 gallon} = \pi r^2 h$$

$$\pi r^2 h = 12705$$
$$h = \frac{12705}{\pi r^2 h}$$

Now, we can create a cost function:

$$C = \frac{34}{144}[2\pi r(h+\frac{13}{8})]+\frac{45}{144}\cdot 2[2(r+\frac{5}{8})]^2+\frac{10}{12}[2\cdot 2\pi(r+\frac{5}{8})]+\frac{2}{12}[2\cdot 2\pi(r+\frac{5}{8})]+\frac{10}{12}(h+\frac{13}{8})$$

Which can be simplified to:

$$C = \frac{17\pi r}{36}(h + \frac{13}{8}) + \frac{5}{2}(r + \frac{5}{8})^2 + 4\pi(r + \frac{5}{8}) + \frac{5}{6}(h + \frac{13}{8})$$

Now we can substitute $\frac{12705}{\pi r^2}$ for h and can define C in terms of r:

$$C = \frac{17\pi r}{36}(\frac{12705}{\pi r^2} + \frac{13}{8}) + \frac{5}{2}(r + \frac{5}{8})^2 + 4\pi(r + \frac{5}{8}) + \frac{5}{6}(\frac{12705}{\pi r^2} + \frac{13}{8})$$

Which can be simplified to:

$$C = \frac{5}{2}r^2 + \frac{221\pi}{288}r + \frac{25}{8}r + 4\pi r + \frac{4235 \cdot 17}{12r} + \frac{4235 \cdot 5}{2\pi r^2} + \frac{125}{128} + \frac{65}{48} + \frac{20\pi}{8}$$

To continue, we must take $\frac{dC}{dr}$ and set it to zero to find the critical points. We must also take the second derivative to determine which critical point is a maximum:

$$\frac{dC}{dr} = 5r + \frac{1373\pi}{288} + \frac{25}{8} - \frac{4235 \cdot 17}{12r^2} - \frac{4235 \cdot 5}{\pi r^3}$$
$$\frac{d^2C}{dr^2} = \frac{4235 \cdot 17}{6r^3} + \frac{3 \cdot 4235 \cdot 5}{\pi r^4}$$

The critical points of $\frac{dC}{dr}$ are at $r = -1.126, 0$, and 9.929. By plugging these numbers into $\frac{d^2C}{dr^2}$, the minimum cost is found to be achieved at $r = 9.929$ inches. We can then plug r into the original function to find h:

$$h = \frac{12705}{\pi r^2}$$
$$h = \frac{12705}{\pi(9.929)^2}$$
$$h = 41.02$$

But due to the way the variables are defined, the height of the optimal cylinder is actually $h + \cdot \frac{5}{8} = 41.02 + \cdot \frac{5}{8} = 42.27$ inches.

So, the optimal dimensions of the cylinder are:

- $r = 9.929$ inches
- $h = 42.27$ inches

However, the actual dimensions of the 55-gallon Tight Head Steel Drum are:

- $r = 12$ inches
- $h = 35$ inches

As you can see, the actual dimensions of the 55-gallon Tight Head Steel Drum are not the most cost effective.

Conclusion

This has been the most difficult project for me to solve all year. I got stumped many times and often got frustrated with the problem. It took a lot of peer collaboration and discussion to finally figure out the solution. While the initial cylinder surface area problem was not too difficult, the cost parameter and dimensions of the 55-gallon Tight Head Steel Drum made the problem very confusing. I found keeping track of all the specifications and dimensions to be very difficult for me.

I honestly don't think that I could have done this project alone. I collaborated with many people to get a final answer. However, the collaboration was very effective and I enjoyed hearing other people's ideas, answers, and opinions.

People I Collaborated With

- Hector Cantu
- Vicki Curtin
- Fletcher Barnhill
- Jackie Smith