Parametric Projectile Project

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February 22, 2017

Abstract

The Problem

A cannon fires a projectile at some angle θ (with respect to the horizontal) with an initial velocity of v_0 and hits a wall that is d meters away. Create a set of parametric equations that models the flight of the projectile, as well as the bounce off of the wall.

Some Rules and Reminders

1. The impact between the projectile and the wall is purely elastic. Because of this, the horizontal velocity is **constant** and stays the same after the projectile hits the wall.

2. Depending on the values of θ, v_0, and d, the projectile may fall short of the wall.

3. Assume that the wall is infinitely tall; the projectile will never clear the wall.

4. θ is measured in radians. v_0 is measured in meters per second. d is measured in meters.

Parametric Equations

Parametric Equations are extremely effective at describing two-dimensional curves by separating the x and y positions. For example, the position, P, of a point at any time t can be written as

$$P : \begin{cases} x &= f(t) \\ y &= g(t) \end{cases}$$

where f and g are functions of t.

Part One – The Cannonball

Values Used

While the parametric equations are applicable to any conditions, I will use my own θ, v_0, and d values for the sake of modeling the functions (d is not applicable until part two). My values are

- $\theta = \pi/3$ radians
- $v_0 = 25$ m/s
- $d = 50$ meters

Gravity

If one was to fire a projectile into the air from ground level with an angle θ and an initial velocity v_0, the parametric equation to model the position of the projectile would be

$$P : \begin{cases} x(t) & = (v_0 \cos\theta)t \\ y(t) & = (v_0 \sin\theta)t \end{cases}$$

While this is a correct parametric equation, gravitational pull has not been taken into account. Without the force of gravity, the projectile will continue indefinitely at the same speed and direction. When graphed, path of the projectile looks like Figure 1:

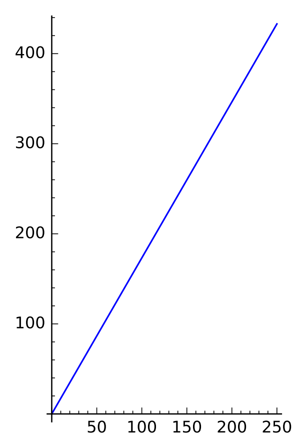

Figure 1: A projectile fired with $\theta = \pi/3$ and $v_0 = 25$ m/s. However, gravitational pull has not been taken into account.

Because gravity only affects the y parameter, the x parameter will remain unchanged when accounting for the gravitational pull. To adjust the y parameter, I used the projectile motion model: $y(t) = -1/2gt^2 + v_0t + y_0$ where g is the gravitational constant of 9.8 meters/second^2 and y_0 is the initial y position. Because both parametric equations and the projectile motion model are functions of time, the projectile motion model can be used in our parametric equation. Therefore, the model is changed to

$$P(t) = \begin{cases} x(t) = v_0 cos(\theta) t \\ y(t) = -4.9t^2 + v_0 sin(\theta) t + y_0 \end{cases}$$

$$0 \le t \le t_f \text{ without wall}$$
$$0 \le t \le t_i \text{ with wall}$$

or

$$P(t) = \begin{cases} x(t) = 25cos(\pi/3)t \\ y(t) = -4.9t^2 + 25sin(\pi/3)t \end{cases}$$

$$0 \le t \le t_f \text{ without wall}$$
$$0 \le t \le t_i \text{ with wall}$$

Where t_f is the time that the projectile hits the ground. Solve for t_f by setting the y parameter of the equation to 0 and solve using the quadratic formula. The non-zero value will be t_f. ***NOTE, t_i is the time that the projectile hits the wall in Part 2. t_f will only be used when there is either no wall or the projectile falls short of the wall.**

The parametric equation without a wall looks like Figure 2 when represented graphically:

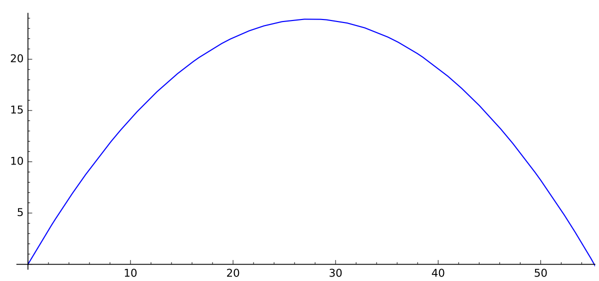

Figure 2: A projectile fired with $\theta = \pi/3$ and $v_0 = 25$ m/s. This time, gravitational pull has been taken into account. In addition, there is no wall, constituting a t range of $0 \le t \le t_f$.

Solving for t_i

This time, assume that there is a wall 50 m away so that we can solve for t_i. To solve for t_i, use the property: time = distance / velocity, or

$$t_i = d/v_0 cos(\theta)$$

After plugging in the values, you get $t_i = 50/25cos(\pi/3) = 4$.

The projectile will hit the wall 4 seconds after launch.

We now have a successful model for projectile motion using parametric functions!

Part Two - Bouncing Off a Wall

Now that we have derived a parametric equation for projectile motion, we must derive a parametric equation that accounts for the projectile bouncing off of a wall.

The Equation:

$$P(t) = \begin{cases} x(t) = d - v_0 cos(-\theta)t \\ y(t) = -4.9t^2 - v_0 sin(-\theta)t + y_w \end{cases}$$

$$0 \leq t \leq t_g$$

where y_w is the y-position of the projectile the instant it comes into contact with the wall and t_g is the time that the projectile hits the ground.

Explanation:

- The t range starts from 0 because the bounce is expressed as a separate parametric equation. Because of this, for the sake of the equation, we assume that time "resets" to 0 after t_i.
- v_0 and θ are negative in both the x and y parameters because the equation works out algebraically when plugging in a t value. For example, when $t = 0$, you will get coordinates of (d, y_w). This is true because at $t = 0$, the projectile has just made impact with the wall.
- The initial x-position of the projectile at the bounce is the same as the position of the wall relative to the cannon, or d. Therefore, $d = x_0$.
- y_w is the y-position of the projectile at the moment of impact with the wall. Because this parametric equation starts right after impact, y_w serves as the initial y-position.

Solving For Unknowns:

After plugging in my values for θ, v_0, and d, you get this parametric equation:

$$P(t) = \begin{cases} x(t) = -25cos(-\pi/3)t + 50 \\ y(t) = -4.9t^2 - 25sin(-\pi/3)t + y_w \end{cases}$$

$$0 \leq t \leq t_g$$

However, there are still two unknowns, y_w , and t_g.

- To solve for y_w, use the concept: $y_w = y(t_i)$. Expanding y_w yields:

$$y_w = -4.9(d/v_0 cos(\theta))^2 + v_0 sin(\theta)(d/v_0 cos(\theta))$$

$$y_w = -4.9(50/25cos(\pi/3))^2 + 25sin(\pi/3)(50/25cos(\pi/3))$$

$$y_w = 8.2$$

The projectile will hit the wall at a height of 8.2 meters above the ground.

- To solve for t_g, set up the y-parameter of the parametric equation where $y = 0$ and solve using the quadratic formula.

$$-4.9t^2 - 25sin(-\pi/3)t = 0$$

$$t = 0, 4.42$$

$$t = 4.42$$

The projectile will hit the ground approximately 4.42 seconds after it bounces off the wall.

When graphed, the entire projectile launch looks like Figure 3:

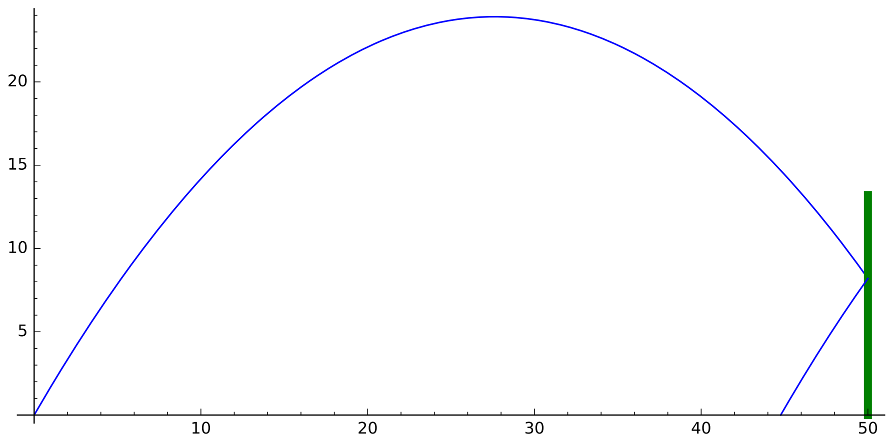

Figure 3: Entire projectile launch, where $\theta = \pi/3$, $v_0 = 25$, and $d = 50$.

Conclusion

The final parametric equations when accounting for gravity are as follows:

Before Impact:

$$P(t) = \begin{cases} x(t) = v_0 cos(\theta)t \\ y(t) = -4.9t^2 + v_0 sin(\theta)t + y_0 \end{cases}$$

$$0 \leq t \leq t_i$$

After Impact:

$$P(t) = \begin{cases} x(t) = d - v_0 cos(-\theta)t \\ y(t) = -4.9t^2 - v_0 sin(-\theta)t + y_w \end{cases}$$

$$0 \leq t \leq t_g$$

Unknowns:

- $y_w = -4.9(d/v_0cos(\theta))^2 + v_0sin(\theta)(d/v_0cos(\theta))$
- $t_i = d/v_0cos(\theta)$
- To solve for t_g, set up the y-parameter of the parametric equation where $y = 0$ and solve using the quadratic formula.

$$-4.9t^2 - v_0sin(-\theta)t = 0$$

Values and Answers:

- $\theta = \pi/3$ radians.
- $v_0 = 25$ m/s.
- $d = 50$ m.
- The projectile will hit the wall 4 seconds after launch.
- The projectile will hit the wall at a height of 8.2 meters above the ground.
- The projectile will hit the ground 4.42 seconds after it bounces off the wall.

This project was extremely interesting, but it was the most challenging one for me so far. I found the most trouble when graphing in SAGE Worksheet. My graph in SAGE Worksheet function differently than my model in Desmos. Luckily, I got Mr. Abell's help and found out why. Otherwise, I enjoy writing in LaTeX and look forward to future projects.

Extension:

As a fun little extension, suppose that when the projectile hit the wall, the gravitational pull was inverted! The projectile would then float in the air. Just imagine the path after the bounce inverted in the x and y axes. Here is a graphic showing the path of this projectile: